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Oriented Circles in Non-Euclidean Space, IV.

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1. The object of this paper is to apply to the geometry of circles the method of the discussion of line geometry in non-euclidean space.¹ As the Study coordinates of circles are analogous to those of the soma of the second sort², the present discussion is also analogous to that of the soma in non-euclidean space which was given by Prof. Nishiuchi³.

In order to carry out this study, we shall confine ourselves to the discussion of such a system of circles that the planes of the circles pass through a fixed point P and that the power of the circles with respect to that point is constant.

Let d be the distance between the fixed point P and the center of a circle of our system whose radius is r , then the power will be

$$\tan \frac{d+r}{2k} \tan \frac{d-r}{2k} = K,$$

where K is constant.

¹ Study, 'Zur nicht euklidischen und linien Geometrie.' Jahresber. D. M. Ver., 11 (1902). Coolidge's dissertation, 'The Dual projective Geometry of Elliptic and Spherical Space,' Greifswald (1902). Beck's dissertation, 'Die Strahlenketten im hyperbolischen Raume,' Hannover (1905).

² Study, Geometrie der Dynamen. Nishiuchi and Kashiwagi. 'Oriented Circles in Non-Euclidean Space,' Mem. Coll. Sci. Kyoto, 4 (1920).

³ 'Geometry of Soma in Non-Euclidean Space,' Mem. Coll. Sci. Kyoto,

When K is positive, we have a fixed sphere orthogonal to circles of our system whose center is at P , and its radius R is given by

$$\cos^2 \frac{r}{k} \cos^2 \frac{R}{k} = \cos^2 \frac{d}{k}.$$

But in this case the radius of the circle which is in bi-involution with another circle will be imaginary, so for the present treatment we may abandon such a case, and take the case only where K is negative.

Here, we can put

$$\frac{\cos \frac{d}{k}}{\cos \frac{r}{k}} = \frac{1-K}{1+K} \left(= \frac{1}{\cos \varphi} \right),$$

where φ is constant.

Suppose that the coordinates of the point P are

$$(k : 0 : 0 : 0),$$

then we have

$$\begin{aligned} a_0 &= k \cos \frac{r}{k} \sec \varphi, & (a \ a) &= k^2, \\ b_0 &= 0, & (b \ b) &= k^2, & (ab) &= 0, \end{aligned}$$

where (a) is the coordinates of the center of a circle and (b) those of the plane of the circle, and r is its radius.

In this case, if we take $(\mathfrak{X})$ as the Study coordinates of the circle, then we have

$$\mathfrak{X}_0 = 0, \quad \mathfrak{X}_j = \sqrt{-1} \frac{b_i}{k} \cos \frac{r}{k},$$

$$\mathfrak{X}_{0i} = \frac{b_i}{k} \cos \frac{r}{k} \sec \varphi,$$

$$\mathfrak{X}_{ij} = \begin{vmatrix} \frac{a_i}{k} & \frac{a_j}{k} \\ \frac{b_i}{k} & \frac{b_j}{k} \end{vmatrix} \quad (i, j = 1, 2, 3).$$

And we easily see that

$$\rho x_{01} \equiv \frac{b_1}{k} \cos \frac{r}{k} \tan \varphi, \quad \rho x_{02} \equiv \frac{b_2}{k} \cos \frac{r}{k} \tan \varphi, \quad \rho x_{03} \equiv \frac{b_3}{k} \cos \frac{r}{k} \tan \varphi,$$

$$\rho x_{23} \equiv \frac{1}{k^2} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad \rho x_{31} \equiv \frac{1}{k^2} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \quad \rho x_{12} \equiv \frac{1}{k^2} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

shall be taken as the coordinates of a circle of our system.

It is evident that there exists the following relation

$$(\mathcal{X} | \mathcal{X}) \equiv \sum x_{ij} x_{kl} = 0.$$

When the circle is null, we have

$$(\mathcal{X} \mathcal{X}) \equiv \sum x_{ij} x_{ij} = 0.$$

Next, suppose that we have two circles of our system whose coordinates are $(\mathcal{X})$ and $(\mathcal{X}')$ respectively, then their mutual power will be proportional to

$$(\mathcal{X} \mathcal{X}'),$$

and the cosine of the intersecting angle θ will be given by

$$\cos \theta = \frac{(\mathcal{X} \mathcal{X}')}{\sqrt{(\mathcal{X} \mathcal{X})} \sqrt{(\mathcal{X}' \mathcal{X}')}}.$$

The condition that they should be in involution is that

$$(\mathcal{X} \mathcal{X}') = 0,$$

and the condition that they should be cospherical is that

$$(\mathcal{X} | \mathcal{X}') = 0.$$

This is also the condition that the two circles intersect at two points.

Their moment and commoment will be

$$\sin \frac{d}{k} \sin \frac{k'}{k} = \frac{\tan \varphi}{k^3 \sin \frac{r}{k} \sin \frac{r'}{k}} \begin{vmatrix} a_1 \cos \frac{r'}{k} - a'_1 \cos \frac{r}{k} & \dots & a_3 \cos \frac{r'}{k} - a'_3 \cos \frac{r}{k} \\ b_1 & \dots & b_3 \\ b'_1 & \dots & b'_3 \end{vmatrix}$$

$$= \frac{(\mathcal{X} | \mathcal{X}')}{\sqrt{(\mathcal{X} \mathcal{X})} \sqrt{(\mathcal{X}' \mathcal{X}')}},$$

$$\cos \frac{d}{k} \cos \frac{d'}{k} = \frac{(\mathcal{X} \mathcal{X}')}{\sqrt{(\mathcal{X} \mathcal{X})} \sqrt{(\mathcal{X}' \mathcal{X}')}}.$$

And their non-intersecting angle θ will be given by

$$(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')\sin^4\theta + [(\mathfrak{X}\mathfrak{X}')^2 - (\mathfrak{X}|\mathfrak{X}')^2 - (\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')]\sin^2\theta + (\mathfrak{X}|\mathfrak{X}')^2 = 0.$$

And the condition that the circles should be paratactic is

$$\begin{aligned} & \{((\mathfrak{X}|\mathfrak{X}') + (\mathfrak{X}\mathfrak{X}'))^2 - (\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')\} \times \\ & \{((\mathfrak{X}|\mathfrak{X}') - (\mathfrak{X}\mathfrak{X}'))^2 - (\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')\} = 0. \end{aligned}$$

The equations

$$\begin{aligned} \rho\mathfrak{Y}_{ij} &= \lambda\mathfrak{X}_{ij} + \mu\mathfrak{X}'_{ij} \\ (i, j &= 0, 1, 2, 3), \end{aligned}$$

will represent a circle when, and only when

$$(\mathfrak{X}|\mathfrak{X}') = 0,$$

i. e. the two circles $(\mathfrak{X})$ and $(\mathfrak{X}')$ are cospherical or intersect at two points, and by varying $\lambda : \mu$ in this case we get all circles coaxial with $(\mathfrak{X})$ and $(\mathfrak{X}')$.

Similarly, when we have three circles $(\mathfrak{X})$, $(\mathfrak{X}')$ and $(\mathfrak{X}'')$, then the equations

$$\begin{aligned} \rho\mathfrak{Y}_{ij} &= \lambda\mathfrak{X}_{ij} + \mu\mathfrak{X}'_{ij} + \nu\mathfrak{X}''_{ij} \\ (i, j &= 0, 1, 2, 3), \end{aligned}$$

will represent a circle when, and only when,

$$(\mathfrak{X}'|\mathfrak{X}'') = (\mathfrak{X}''|\mathfrak{X}) = (\mathfrak{X}|\mathfrak{X}') = 0,$$

i. e. two by two they are cospherical, and in this case either all pass through two points, or are on one sphere.

Next, we shall find a circle of our system which is in bi-involution with the circle $(\mathfrak{X})$.

Let $(\mathfrak{X}')$ be the coordinates of the required circle, then we have

$$a'_0 = \lambda a_0 + \mu k = k \cos \frac{r'}{k} \sec \varphi, \quad a'_i = \lambda a_i,$$

$$b'_0 = 0, \quad b'_i = \nu \frac{\partial}{\partial t} |t \ a \ b|$$

$$(i = 1, 2, 3),$$

where

$$\lambda = \frac{-\cos \frac{r}{k} \cos \frac{r'}{k} \sin^2 \varphi}{\cos^2 \varphi - \cos^2 \frac{r}{k}}, \quad \mu = \frac{\sin^2 \frac{r}{k} \cos \frac{r'}{k} \cos \varphi}{\cos^2 \varphi - \cos^2 \frac{r}{k}},$$

$$\nu = \frac{\cos \varphi}{k \sqrt{\cos^2 \varphi - \cos^2 \frac{r}{k}}}, \quad \cos^2 \frac{r'}{k} = \frac{\cos^2 \varphi - \cos^2 \frac{r}{k}}{\sin^2 \frac{r}{k} - \cos^2 \frac{r}{k} \sin^2 \varphi}.$$

So we have

$$\rho' x'_{0i} = \frac{b'_i \cos \frac{r'}{k} \tan \varphi}{k} = x_{jk} \frac{\sin \varphi}{\sqrt{\sin^2 \frac{r}{k} - \cos^2 \frac{r}{k} \sin^2 \varphi}},$$

$$\rho' x'_{jk} = \frac{1}{k^2} \begin{vmatrix} a'_j & a'_k \\ b'_j & b'_k \end{vmatrix} = x_{0i} \frac{\sin \varphi}{\sqrt{\sin^2 \frac{r}{k} - \cos^2 \frac{r}{k} \sin^2 \varphi}}.$$

2. We shall define as a *complex* of circles of our systems one, the coordinates of each member of which are proportional to analytic functions of three independent variables, the ratios not being all functions of two variables.

According to this definition, a general complex may be written by the equation

$$f(\mathcal{X}) = 0.$$

If $f(\mathcal{X})$ be linear, the complex is said to be *linear* and the simplest linear complex may be given by

$$(\mathcal{A}\mathcal{X}) \equiv \sum \mathcal{A}_{ij} \mathcal{X}_{ij} = 0.$$

If

$$(\mathcal{A}|\mathcal{A}) = 0,$$

the complex consists in the totality of circles cospherical with the fixed circle whose coordinates $(\mathcal{Y})$ are

$$\mathcal{Y}_{ij} = \rho \mathcal{A}_{ij}.$$

A system of circles whose coordinates are proportional to analytic functions of two independent variables, their ratios not being all functions of one variable, shall be called a *congruence*. It is evident that

our circles may form a focal congruence¹ and it may be given by the two equations

$$f(\mathcal{X})=0, \quad \varphi(\mathcal{X})=0.$$

Circles of our system which are in bi-involution with those of the given complex

$$(\mathfrak{A}\mathcal{X})=0,$$

will generate a second complex

$$(\mathfrak{A}|\mathcal{X})=0.$$

The focal congruence, whose equations are

$$(\mathfrak{A}\mathcal{X})=(\mathfrak{A}|\mathcal{X})=0,$$

will be composed of all circles of our complex and those which are in bi-involution with them, or common to all complexes

$$l(\mathfrak{A}\mathcal{X})+m(\mathfrak{A}|\mathcal{X})=0.$$

These complexes shall be said to form a *coaxal system*.

In the coaxal system, there are two special complexes for which $l:m$ is given by the relation

$$\sum (l\mathfrak{A}_{ij}+m\mathfrak{A}_{kl})(l\mathfrak{A}_{kl}+m\mathfrak{A}_{ij})=0,$$

i. e.

$$(l^2+m^2)(\mathfrak{A}|\mathfrak{A})+2lm(\mathfrak{A}\mathfrak{A})=0.$$

Let the two values of l/m be l_1/m_1 , l_2/m_2 , then

$$\frac{l_1}{m_1}=-\mathcal{J}+\sqrt{\mathcal{J}^2-1}, \quad \frac{l_2}{m_2}=-\mathcal{J}-\sqrt{\mathcal{J}^2-1}, \quad \mathcal{J}=\frac{(\mathfrak{A}\mathfrak{A})}{(\mathfrak{A}|\mathfrak{A})},$$

and the two special complexes are given by

$$l_1(\mathfrak{A}\mathcal{X})+m_1(\mathfrak{A}|\mathcal{X})=0,$$

$$l_2(\mathfrak{A}\mathcal{X})+m_2(\mathfrak{A}|\mathcal{X})=0,$$

respectively.

If

$$\mathcal{J} \neq 1,$$

then the two are distinct and each of the complexes will be composed

¹ Kashiwagi, Mem. Col. Sci. Kyoto, 5, 383, 6, 97.

of all circles common to the two complexes

$$(\mathfrak{A}\mathfrak{X})=0, (\mathfrak{A}|\mathfrak{X})=0,$$

and consists in the totality of circles cospherical with the fixed circles

$$\rho\alpha_{ij} = \left(-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}\right)\mathfrak{A}_{ij} + \mathfrak{A}_{kl},$$

$$\rho\alpha'_{ij} = \mathfrak{A}_{ij} + \left(-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}\right)\mathfrak{A}_{kl},$$

respectively.

It is immediately evident that these two are in bi-involution.

Let $(\mathfrak{X})$ be any circle of our system, then the ratio of the moment and commoment of the circle with regard to those two circles will be

$$\frac{\sin \frac{d_1}{k} \sin \frac{d_2}{k}}{\sin \frac{d'_1}{k} \sin \frac{d'_2}{k}} = \frac{\left(-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}\right)(\mathfrak{A}|\mathfrak{X}) + (\mathfrak{A}\mathfrak{X})}{(\mathfrak{A}|\mathfrak{X}) + \left(-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}\right)(\mathfrak{A}\mathfrak{X})},$$

$$\frac{\cos \frac{d_1}{k} \cos \frac{d_2}{k}}{\cos \frac{d'_1}{k} \cos \frac{d'_2}{k}} = \frac{\left(-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}\right)(\mathfrak{A}\mathfrak{X}) + (\mathfrak{A}|\mathfrak{X})}{(\mathfrak{A}\mathfrak{X}) + \left(-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}\right)(\mathfrak{A}|\mathfrak{X})},$$

respectively. For a circle of our complex

$$(\mathfrak{A}\mathfrak{X})=0,$$

these ratios will be reduced to

$$-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1},$$

and

$$-\mathfrak{J} - \sqrt{\mathfrak{J}^2 - 1},$$

respectively.

Conversely, when one of these ratios is reduced to $-\mathfrak{J} + \sqrt{\mathfrak{J}^2 - 1}$ or $-\mathfrak{J} - \sqrt{\mathfrak{J}^2 - 1}$ the circle $(\mathfrak{X})$ will satisfy the relation

$$(\mathfrak{A}\mathfrak{X})=0,$$

that is, the circle belongs to our complex.

A pair of real circles which are in bi-involution with each other shall be said to form a *propre cross*, and the cross formed by these two circles (α) and (α') shall be the *axial cross* of the complex.

A linear complex where (α) and (α') are distinct, i. e.

$$\mathfrak{J} \neq 1$$

is said to be *general*.

CIRCLE CROSSES IN HYPERBOLIC SPACE.

3. We shall now continue the discussion of our systems of circles in the special direction where the fundamental element is not, in general, a circle, but a pair of circles (circle cross) invariantly connected.

Let us start in the real domain of hyperbolic space, and for the sake of brevity, we put the measure of curvature of space $1/k^2$ equal to -1 , and take as the coordinates of a point (x) four numbers $(\dot{x})$ such that

$$(\dot{x}) = \dot{x}_0, \dot{x}_1, \dot{x}_2, \dot{x}_3, \\ -\dot{x}_0^2 + \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = -1.$$

The equation to the absolute may be written in the form

$$\dot{x}_0^2 - \dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2 = 0$$

And for the coordinates of our circle, we can take the following real numbers

$$\lambda \dot{x}_{0i} = \dot{b}_i \cosh r \tan \varphi,$$

$$\lambda \dot{x}_{kl} = - \begin{vmatrix} \dot{a}_k & \dot{a}_l \\ \dot{b}_k & \dot{b}_l \end{vmatrix}$$

$$(i, k, l = 1, 2, 3., i \neq k \neq l).$$

If $(\dot{x}')$ be the coordinates of our circle which is in bi-involution with the given circle $(\dot{x})$, then

$$\dot{x}'_{0i} = \mu \dot{x}_{kl},$$

$$\dot{x}'_{kl} = -\mu \dot{x}_{0i}.$$

Next, we shall consider a linear complex whose equation is

$$(\dot{\mathfrak{U}}|\dot{\mathfrak{X}}) \equiv \sum \dot{\mathfrak{U}}_{0i} \dot{\mathfrak{X}}_{0i} - \sum \dot{\mathfrak{U}}_{jk} \dot{\mathfrak{X}}_{jk} = 0.$$

Then the coaxal system will be given by

$$l(\dot{\mathfrak{U}}|\dot{\mathfrak{X}}) + m(\dot{\mathfrak{U}}|\dot{\mathfrak{X}}) = 0$$

and the coordinates of the circle of the axial cross will have the form

$$\dot{\mathfrak{X}}_{0i} = l\dot{\mathfrak{U}}_{0i} - m\dot{\mathfrak{U}}_{jk}, \quad \dot{\mathfrak{X}}_{jk} = l\dot{\mathfrak{U}}_{jk} + m\dot{\mathfrak{U}}_{0i}.$$

Let us now write

$$\begin{aligned} \dot{\mathfrak{U}}_{01} + \sqrt{-1} \dot{\mathfrak{U}}_{23} &= \rho X_1, \\ \dot{\mathfrak{U}}_{02} + \sqrt{-1} \dot{\mathfrak{U}}_{31} &= \rho X_2, \\ \dot{\mathfrak{U}}_{03} + \sqrt{-1} \dot{\mathfrak{U}}_{12} &= \rho X_3. \end{aligned} \quad (1)$$

If we replace (X) by $l(X) + im(X)$, we get

$$\rho' X'_i = (l\dot{\mathfrak{U}}_{0i} - m\dot{\mathfrak{U}}_{jk}) + \sqrt{-1} (l\dot{\mathfrak{U}}_{jk} + m\dot{\mathfrak{U}}_{0i}),$$

and the new linear complex

$$(\dot{\mathfrak{U}}'|\dot{\mathfrak{X}}) = 0$$

where

$$\dot{\mathfrak{U}}'_{0i} = l\dot{\mathfrak{U}}_{0i} - m\dot{\mathfrak{U}}_{jk}, \quad \dot{\mathfrak{U}}'_{jk} = l\dot{\mathfrak{U}}_{jk} + m\dot{\mathfrak{U}}_{0i},$$

will be a coaxal system and have the same axial cross as the original one.

If either of the circles of a cross have the coordinates $(\dot{\mathfrak{U}})$ then the three numbers (X) given by equation (1) may be taken to represent the cross.

Suppose that we have a triad of coordinates (X) , then the coordinates of the corresponding cross will be found from (1) by assuming for ρ such a value that the coordinates $(\dot{\mathfrak{U}})$ shall satisfy the identity

$$(\dot{\mathfrak{U}}|\dot{\mathfrak{U}}) = 0.$$

For that it is necessary and sufficient that the imaginary part of $\rho^2(XX)$ should vanish, i.e.

$$\begin{aligned}\sigma\dot{\alpha}_{0i} &= \left[\frac{X_i}{\sqrt{(XX)}} + \frac{\bar{X}_i}{\sqrt{(\bar{X}\bar{X})}} \right], \\ \sigma\dot{\alpha}_{jk} &= \sqrt{-1} \left[\frac{X_i}{\sqrt{(XX)}} - \frac{\bar{X}_i}{\sqrt{(\bar{X}\bar{X})}} \right].\end{aligned}\tag{2}$$

with the condition

$$(XX) \neq 0, \quad (\bar{X}\bar{X}) \neq 0.$$

Let the circles of a cross have the coordinates $(\dot{\alpha})$ and $(\dot{\alpha}')$, where

$$\begin{aligned}\dot{\alpha}_{0i} &= -\lambda \dot{\alpha}'_{jk}, \\ \dot{\alpha}_{jk} &= \lambda \dot{\alpha}'_{0i}.\end{aligned}$$

In the same way let a second cross be determined by the circles $(\dot{\beta})$ and $(\dot{\beta}')$. The condition that the circles $(\dot{\alpha})$ and $(\dot{\beta})$ should be cospherical, or that one should be cospherical with the mate of the other in a circle cross, is that

$$(\dot{\alpha}|\dot{\beta})=0,$$

i.e.

$$\frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}} = \pm \frac{(\bar{X}\bar{Y})}{\sqrt{(\bar{X}\bar{X})}\sqrt{(\bar{Y}\bar{Y})}}.\tag{3}$$

They will be cospherical and in involution, if

$$(XY) = (\bar{X}\bar{Y}) = 0.\tag{4}$$

We shall say that two crosses intersect orthogonally if their circles intersect and are in involution.

If the circle (α) be cospherical with the two circles (β) and (β') of a certain cross, then

$$\begin{aligned}(\dot{\alpha}|\dot{\beta}) &= (\dot{\alpha}|\dot{\beta}') = 0, \\ (\dot{\alpha}'|\dot{\beta}) &= (\dot{\alpha}'|\dot{\beta}') = 0.\end{aligned}$$

And we have the following:

If a not null circle be cospherical with two circles of a cross, it will cut each of them orthogonally and in involution with them.

Let θ_1, θ_2 and d_1, d_2 be the angles and the distances of the circles

$(\dot{\alpha})$ and $(\dot{\beta})$ respectively, then their commoment and moment will be

$$\begin{aligned}
 -\cosh d_1 \cosh d_2 &= -\cos \theta_1 \cos \theta_2 = \frac{(\dot{\alpha}|\dot{\beta})}{\sqrt{(\dot{\alpha}|\dot{\alpha})}\sqrt{(\dot{\beta}|\dot{\beta})}} \\
 &= \frac{1}{2} \left[\frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}} + \frac{(\bar{X}\bar{Y})}{\sqrt{(\bar{X}\bar{X})}\sqrt{(\bar{Y}\bar{Y})}} \right], \\
 \sinh d_1 \sinh d_2 &= -\sin \theta_1 \sin \theta_2 = \frac{\sqrt{1-i(\dot{\alpha}|\dot{\beta})}}{\sqrt{(\dot{\alpha}|\dot{\alpha})}\sqrt{(\dot{\beta}|\dot{\beta})}} \\
 &= \frac{1}{2} \left[\frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}} - \frac{(\bar{X}\bar{Y})}{\sqrt{(\bar{X}\bar{X})}\sqrt{(\bar{Y}\bar{Y})}} \right].
 \end{aligned} \tag{5}$$

Here, we can establish one to one correspondence between the totality of all circle crosses in hyperbolic space and the totality of all complex points (or lines) in an elliptic plane. Two intersecting crosses will correspond to points, and crosses intersecting orthogonally will correspond to orthogonal points of the elliptic plane.

The totality of crosses (U) which intersect a given cross (X) orthogonally will be given by means of the linear equation

$$(UX)=0.$$

A linear equation will be transformed linearly into another linear one, if the variables and coefficients be treated contragrediently. Now, for the sake of clearness, we shall assume our cross space is doubly overlaid. We shall say that a cross belongs to the *upper layer*, when it is represented by a complex line in the complex plane; when it is represented by a point in that plane, we shall speak of a cross of the *lower layer*.

The necessary and sufficient condition that two crosses of different layers should intersect orthogonally is that the corresponding point and line of the complex plane should be in united position. Thus two crosses of different layers which intersect orthogonally to each other may be represented by a complex line element (system of a point and a line through this point) after Lie.

4. The simplest one-dimensional family of crosses is the *circle-chain* composed of all crosses whose coordinates are linearly dependent, by means of real coefficients, on those of two given crosses, i. e.

$$\rho X_i = a Y_i + b Z_i, \quad (6)$$

$$(i=1, 2, 3).$$

In the representing complex plane, we see that we have ∞^1 points of a line so related that the cross ratio of any four is real.

All crosses of the circle-chain will cut orthogonally by another cross (of the other layer) called the *axis* of the chain. Let (U) be the coordinates of the axis of the chain, then we have

$$(UY)=0,$$

$$(UZ)=0$$

and

$$U_i = \frac{\partial}{\partial t_i} \begin{vmatrix} t & Y & Z \end{vmatrix}.$$

Every circle-chain has at least one pair of real crosses, called the *principal crosses* of the chain, which intersect orthogonally. To find these, we must solve the equation

$$aa'(YY) + (ab' + a'b)(YZ) + bb'(ZZ) = 0,$$

$$aa'(\bar{Y}\bar{Y}) + (ab' + a'b)(\bar{Y}\bar{Z}) + bb'(\bar{Z}\bar{Z}) = 0.$$

By the elimination of $a' : b'$ from these equations, we have a quadratic in $a : b$; i.e.

$$a^2 \begin{vmatrix} (YY) & (YZ) \\ (\bar{Y}\bar{Y}) & (\bar{Y}\bar{Z}) \end{vmatrix} + ab \begin{vmatrix} (YY) & (ZZ) \\ (\bar{Y}\bar{Y}) & (\bar{Z}\bar{Z}) \end{vmatrix} + b^2 \begin{vmatrix} (YZ) & (ZZ) \\ (\bar{Y}\bar{Z}) & (\bar{Z}\bar{Z}) \end{vmatrix} = 0.$$

It is easily seen that this equation has real and distinct roots. If we take the principal crosses as $(1, 0, 0)$, $(0, 1, 0)$, we can express our chain in the simple form

$$X_1 = a(p + qi), \quad X_2 = b(r + si), \quad X_3 = 0, \quad (i = \sqrt{-1}). \quad (7)$$

Eliminating a/b , we get

$$(p\dot{\mathfrak{A}}_{31} + q\dot{\mathfrak{A}}_{02})(r\dot{\mathfrak{A}}_{10} - s\dot{\mathfrak{A}}_{23}) = (p\dot{\mathfrak{A}}_{02} - q\dot{\mathfrak{A}}_{31})(r\dot{\mathfrak{A}}_{23} + s\dot{\mathfrak{A}}_{01}).$$

Let $(\dot{x})$ be the coordinates of a point on the circle of our system, then we have

$$-\dot{b}_0\dot{x}_0+\dot{b}_1\dot{x}_1+\dot{b}_2\dot{x}_2+\dot{b}_3\dot{x}_3=0,$$

Or

$$(\dot{a}\dot{x})=-\cosh r.$$

$$\lambda\dot{x}_{01}=\dot{x}_2\dot{x}_3 \sin\varphi \quad \lambda\dot{x}_{02}=-\dot{x}_1\dot{x}_3 \sin\varphi \quad \lambda\dot{x}_{03}=0,$$

$$\lambda\dot{x}_{23}=\dot{x}_1(\dot{x}_0-\cos\varphi), \quad \lambda\dot{x}_{31}=\dot{x}_2(\dot{x}_0-\cos\varphi), \quad \lambda\dot{x}_{12}=0.$$

Hence, the equation of the chain surface in point coordinates will be

$$(ps-qr)\left[-(\dot{x}_0-\cos\varphi)^2+\dot{x}_3^2\sin^2\varphi\right]\dot{x}_1\dot{x}_2 \tag{8}$$

$$+(pr+qs)(\dot{x}_1^2+\dot{x}_2^2)(\dot{x}_0-\cos\varphi)\dot{x}_3 \sin\varphi=0,$$

where

$$(\dot{x}\dot{x})=-1.$$

If

$$(ps-qr)=0, \text{ or } (pr+qs)=0,$$

we have two real and two imaginary circle systems.

Next, suppose that we have a focal congruence of circles of such a nature that the corresponding cross coordinates are analytic functions of two real parameters u, v , then the cross cutting orthogonally the adjacent crosses (U) and $(U+dU)$ will be determined by the equation

$$X_i=\left|\begin{array}{cc} U_j & U_k \\ \frac{\partial U_j}{\partial u} & \frac{\partial U_k}{\partial u} \end{array}\right| du + \left|\begin{array}{cc} U_j & U_k \\ \frac{\partial U_j}{\partial v} & \frac{\partial U_k}{\partial v} \end{array}\right| dv. \tag{9}$$

If

$$\left|U \frac{\partial U}{\partial u} \frac{\partial U}{\partial v}\right| \equiv 0, \tag{10}$$

there is but one cross cutting orthogonally.

If we take u and v as the focal parameters, then we have

$$\frac{\partial \dot{b}_i}{\partial u} = \lambda \dot{a}_i \sinh r \frac{\partial r}{\partial u} + \lambda \frac{\partial \dot{a}_i}{\partial u} \cosh r + \mu \dot{b}_i,$$

$$\frac{\partial \dot{b}_i}{\partial v} = \lambda' \dot{a}_i \sinh r \frac{\partial r}{\partial v} + \lambda' \frac{\partial \dot{a}_i}{\partial v} \cosh r + \mu' \dot{b}_i,^{(1)}$$

$$(i=0, 1, 2, 3)$$

and from the identity (10)

$$l \frac{\partial \dot{b}_i}{\partial u} + m \frac{\partial \dot{b}_i}{\partial v} = 0,$$

$$p \dot{b}_i = l \frac{\partial \dot{a}_i}{\partial u} + m \frac{\partial \dot{a}_i}{\partial v}.$$

Hence, we get

$$l(\lambda - \lambda') \left(\dot{a} \frac{\partial \dot{b}}{\partial u} \right) \cosh r = 0,$$

i.e.

$$\lambda = \lambda',$$

or

$$\left(\dot{a} \frac{\partial \dot{b}}{\partial u} \right) = \left(\dot{a} \frac{\partial \dot{b}}{\partial v} \right) = 0.$$

But from the last condition, we have

$$\lambda - \lambda' = 0, \text{ or } r = \text{const.}$$

Therefore, in this case, the focal congruence is pseudo-normal and the space axis of the circles of the congruence is a normal one, or the radius of the circles of the congruence is constant.

Let us exclude this case and pass over to the other, where

$$\left| U \frac{\partial U}{\partial u} \frac{\partial V}{\partial v} \right| \neq 0.$$

In this case we shall call the cross in *general position* in our congruence. Hence, we have the following:

The crosses cutting orthogonally a cross in general position in a focal congruence, and each adjacent one, will generate a chain.

The simplest two-dimensional system of circle crosses is the *circle chain congruence* whose coordinates (X) are linearly dependent on those of the crosses (Y), (Z), (T) with real coefficients a , b , c , i.e.

⁽¹⁾ Kashiwagi, Mem. Coll. Sci., 6, 109 (1923).

$$X_i = aY_i + bZ_i + cT_i, \quad (11)$$

$$(i=1, 2, 3).$$

(a)

$$|Y \ Z \ T| \neq 0.$$

In this case the chain congruence is represented by all points of the real domain of the representing plane.

The cross intersecting orthogonally pairs of crosses of the chain congruence will generate a second congruence of like sort in the other layer called the *conjugate* to the first.

The conjugate to the chain congruence (11) will have the equation

$$U_i = p \begin{vmatrix} Y_j & Y_k \\ Z & Y_k \end{vmatrix} + q \begin{vmatrix} Z_j & Z_k \\ T & T_k \end{vmatrix} + r \begin{vmatrix} T_j & T_k \\ Y_j & Y_k \end{vmatrix}.$$

The relation between the two is reciprocal. We can prove that there are three crosses in a chain congruence which belong at once to both systems. Let their coordinates be $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, then the equations of our chain congruence may thus be reduced to the simple form

$$X_1 = a(p + qi), \ X_2 = b(r + si), \ X_3 = c(t + ri).$$

where a, b, c are real homogeneous variables and $i = \sqrt{-1}$.

(b)

$$|Y \ Z \ T| = 0 \quad (12)$$

but $(Y), (Z), (T)$ are not linearly dependent on real coefficients λ, μ, ν , i.e.

$$\lambda Y_i + \mu Z_i + \nu T_i \neq 0.$$

In this case, we have a cross (U) which cuts orthogonally all crosses of the system, where

$$U_i = \begin{vmatrix} Y_j & K_k \\ Z & Z_k \end{vmatrix}.$$

Conversely we can prove that every cross which cuts orthogonally the cross (U) may be expressed in this form.

CIRCLE CROSSES IN ELLIPTIC SPACE.

5. As we relate the geometry of the circle cross in hyperbolic space with that of a point in the complex plane, so we may relate a circle cross in elliptic space to a pair of real points in two planes.

First, let us choose such a unit of measure that $k=1$.

We have for absolute

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,$$

and the coordinates of a circle of our system will be

$$\rho \mathfrak{X}_{01} = b_1 \cos r \tan \varphi, \quad \rho \mathfrak{X}_{02} = b_2 \cos r \tan \varphi, \quad \rho \mathfrak{X}_{03} = b_3 \cos r \tan \varphi,$$

$$\rho \mathfrak{X}_{23} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad \rho \mathfrak{X}_{31} = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \quad \rho \mathfrak{X}_{12} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Next consider a linear complex whose equation is

$$\sum a_{ij} \mathfrak{X}_{ij} = 0$$

and put

$$\mathfrak{U}_{01} + \mathfrak{U}_{23} = \rho_l X_1, \quad \mathfrak{U}_{01} - \mathfrak{U}_{23} = \sigma_r X_1,$$

$$\mathfrak{U}_{02} + \mathfrak{U}_{31} = \rho_l X_2, \quad \mathfrak{U}_{02} - \mathfrak{U}_{31} = \sigma_r X_2,$$

$$\mathfrak{U}_{03} + \mathfrak{U}_{12} = \rho_l X_3, \quad \mathfrak{U}_{03} - \mathfrak{U}_{12} = \sigma_r X_3.$$

If we replace our complex by another coaxal therewith we shall merely multiply $(_l X)$, $(_r X)$ by two different constants. When we wish to move back from independent sets of coordinates $(_l X)$, $(_r X)$ to the degenerate complexes of the system, i.e. to the circles of the cross defined thereby, we have to take for ρ and σ such values that

$$(\mathfrak{U} | \mathfrak{U}) = 0.$$

For this, it is necessary and sufficient that

$$\rho(_l X_l X) - \sigma(_r X_r X) = 0.$$

And we have

$$\tau \mathfrak{U}_{0i} = {}_l X_i \sqrt{(_r X_r X)} + {}_r X_i \sqrt{(_l X_l X)},$$

$$\tau X_{it} = {}_iX_i \sqrt{({}_rX_rX)} - {}_rX_i \sqrt{({}_iX_iX)}.$$

Thus we see that $({}_iX)$, $({}_rX)$ may be taken as two separate triads of coordinates to determine the cross.

Let the circles of the cross have the coordinates (α) and (α') ; then we have

$$\alpha_{ij} = \rho \alpha'_{kl}.$$

And let (β) , (β') be the coordinates of circles of the other cross and let θ , θ' ; d , d' be the angles and distances of the circles (α) and (β) ; then,

$$\cos \theta \cos \theta' = \cos d \cos d' = \frac{(\alpha\beta)}{\sqrt{(\alpha\alpha)}\sqrt{(\beta\beta)}},$$

$$\sin \theta \sin \theta' = \sin d \sin d' = \frac{(\alpha|\beta)}{\sqrt{(\alpha\alpha)}\sqrt{(\beta\beta)}},$$

Hence, we easily find

$$\cos(d+d') = \frac{({}_rX_rY)}{\sqrt{({}_rX_rX)}\sqrt{({}_rY_rY)}}, \quad \cos(d-d') = \frac{({}_iX_iY)}{\sqrt{({}_iX_iX)}\sqrt{({}_iY_iY)}},$$

or else

$$\cos(d+d') = \frac{({}_iX_iY)}{\sqrt{({}_iX_iX)}\sqrt{({}_iY_iY)}}, \quad \cos(d-d') = \frac{(X_rY_r)}{\sqrt{({}_rX_rX)}\sqrt{({}_rY_rY)}},$$

The quantities ${}_iD$ and ${}_rD$ which are given by the equation

$$\cos {}_iD = \frac{({}_iX_iY)}{\sqrt{({}_iX_iX)}\sqrt{({}_iY_iY)}},$$

$$\cos {}_rD = \frac{(X_rY_r)}{\sqrt{({}_rX_rX)}\sqrt{({}_rY_rY)}},$$

shall be called *left and right distances* of the two crosses respectively. These are the algebraic sum of their moment and commoment and are analogous to the Clifford angles of two line crosses in elliptic space.

The condition that the circles (α) and (β) should be cospherical, or that one should be cospherical with the mate of the other in a circle cross, is that

$$\cos_l D = \pm \cos_r D$$

i.e.

$$\frac{(lX_l Y)}{\sqrt{(lX_l X)}\sqrt{(lY_l Y)}} = \pm \frac{(rX_r Y)}{\sqrt{(rX_r X)}\sqrt{(rY_r Y)'}}$$

They will be cospherical and in involution, i.e. the two crosses cut orthogonally, if

$$\cos_l D = \cos_r D = 0$$

i.e.

$$(lX_l Y) = (X_r Y_r) = 0.$$

The condition for parataxy will be

$$\sin_l D \sin_r D = 0,$$

i.e.

$$\left[(lX_l X)(lY_l Y) - (lX_l Y)^2 \right] \cdot \left[(rX_r X)(rY_r Y) - (rX_r Y)^2 \right] = 0.$$

This may be written

$$\left\| \begin{matrix} lX_1 & lX_2 & lX_3 \\ lY_1 & lY_2 & lY_3 \end{matrix} \right\|^2 \cdot \left\| \begin{matrix} rX_1 & rX_2 & rX_3 \\ rY_1 & rY_2 & rY_3 \end{matrix} \right\|^2 = 0.$$

The only real solutions will be

$$lY_i = \rho_l X_i, \quad rY_i = \sigma_r X_i.$$

We can establish one to one correspondence between the totality of all real crosses in elliptic space and the totality of all pairs of points (or lines), one in each of two real planes.

As in the hyperbolic case, so here, we shall look upon cross space as doubly overlaid, and assign a cross to the upper layer if it be determined by two points in the representing planes, while it shall be assigned to the lower layer if it be determined by two lines. Hence, we have the following:

In order that two crosses of different layers should intersect orthogonally, it is necessary and sufficient that they should be represented by line elements in the two planes.

6. The one-dimensional circle chain has the equations

$$\begin{aligned} {}_iX_i &= a_i Y_i + b_i Z_i, & {}_rX_i &= a_r Y_i + b_r Z_i \\ (a, b : \text{real}, \quad i &= 1, 2, 3). \end{aligned}$$

Let us consider the case in which

$${}_iY_i \equiv \rho_i Z_i, \quad {}_rY_i \equiv \sigma_r Z_i.$$

A fixed cross whose equations are

$$\lambda_i U_i \equiv \begin{vmatrix} {}_iY_j & {}_iZ_j \\ {}_iY_k & {}_iZ_k \end{vmatrix}, \quad \mu_r U_i \equiv \begin{vmatrix} {}_rY_j & {}_rZ_j \\ {}_rY_k & {}_rZ_k \end{vmatrix},$$

cuts orthogonally all the crosses of the system considered.

Suppose that we have a focal congruence of circles of our system. We may express them parametrically

$$\begin{aligned} {}_iX_i &\equiv {}_iX_i(u, v), \\ {}_rX_i &\equiv {}_rX_i(u, v). \end{aligned}$$

Let us assume that

$$\left| {}_iX \frac{\partial {}_iX}{\partial u} \frac{\partial {}_iX}{\partial v} \right| \times \left| {}_rX \frac{\partial {}_rX}{\partial u} \frac{\partial {}_rX}{\partial v} \right| \equiv 0.$$

The cross cutting orthogonally the adjacent crosses $({}_iX), ({}_rX)$ and $({}_iX + d{}_iX), ({}_rX + d{}_rX)$ will be determined by the equations

$$\begin{aligned} \rho_i W_i &\equiv \begin{vmatrix} {}_iX_j & {}_iX_k \\ \frac{\partial {}_iX_j}{\partial u} & \frac{\partial {}_iX_k}{\partial u} \end{vmatrix} du + \begin{vmatrix} {}_iX_j & {}_iX_k \\ \frac{\partial {}_iX_j}{\partial v} & \frac{\partial {}_iX_k}{\partial v} \end{vmatrix} dv, \\ \sigma_r W_i &= \begin{vmatrix} {}_rX_j & {}_rX_k \\ \frac{\partial {}_rX_j}{\partial u} & \frac{\partial {}_rX_k}{\partial u} \end{vmatrix} du + \begin{vmatrix} {}_rX_j & {}_rX_k \\ \frac{\partial {}_rX_j}{\partial v} & \frac{\partial {}_rX_k}{\partial v} \end{vmatrix} dv. \end{aligned}$$

We shall mean by the *general position* of a circle in our congruence, one where

$$\left| {}_iX \frac{\partial {}_iX}{\partial u} \frac{\partial {}_iX}{\partial v} \right| \times \left| {}_rX \frac{\partial {}_rX}{\partial u} \frac{\partial {}_rX}{\partial v} \right| \equiv 0.$$

In the general position, if the crosses $(_iX)$, $(_rX)$ and $(_iX+d_iX)$, $(_rX+d_rX)$ are fixed, then the totality of the orthogonal crosses to these crosses forms a circle chain.

Specially, in the case

$$\left| {}_iX \frac{\partial {}_iX}{\partial u} \frac{\partial {}_iX}{\partial v} \right| = 0, \quad \left| {}_rX \frac{\partial {}_rX}{\partial u} \frac{\partial {}_rX}{\partial v} \right| = 0,$$

the cross whose coordinates are

$$\rho_i W_i \equiv \begin{vmatrix} \frac{\partial {}_iX_j}{\partial u} & \frac{\partial {}_iX_k}{\partial u} \\ \frac{\partial {}_iX_j}{\partial v} & \frac{\partial {}_iX_k}{\partial v} \end{vmatrix}, \quad \rho_r W_r \equiv \begin{vmatrix} \frac{\partial {}_rX_j}{\partial u} & \frac{\partial {}_rX_k}{\partial u} \\ \frac{\partial {}_rX_j}{\partial v} & \frac{\partial {}_rX_k}{\partial v} \end{vmatrix}$$

is the common orthogonal cross to a cross $(_iX)$, $(_rX)$ and every one of the crosses adjacent to it.

As in hyperbolic space, there is but one common perpendicular to a cross and its adjacent crosses. And the congruence is pseudo-normal or the radius of the circle of the congruence is constant.

Also, in the elliptic case, every circle chain has at least one pair of real crosses, called *principal crosses*, which are orthogonal. To find these, we must solve the equation

$$\begin{aligned} aa'({}_iY_iY) + (ab' + b'a)({}_iY_iZ) + bb'({}_iZ_iZ) &= 0, \\ aa'({}_rY_rY) + (ab' + b'a)({}_rY_rZ) + bb'({}_rZ_rZ) &= 0. \end{aligned}$$

Eliminating $a' : b'$, we get

$$a^2 \left| \begin{pmatrix} {}_iY_iY \\ {}_rY_rY \end{pmatrix} \begin{pmatrix} {}_iY_iZ \\ {}_rY_rZ \end{pmatrix} \right| + ab \left| \begin{pmatrix} {}_iY_iY \\ {}_rY_rY \end{pmatrix} \begin{pmatrix} {}_iZ_iZ \\ {}_rZ_rZ \end{pmatrix} \right| + b^2 \left| \begin{pmatrix} {}_iY_iZ \\ {}_rY_rZ \end{pmatrix} \begin{pmatrix} {}_iZ_iZ \\ {}_rZ_rZ \end{pmatrix} \right| = 0.$$

The discriminant of the equation is

$$\left| \begin{pmatrix} {}_iY_iY \\ {}_rY_rY \end{pmatrix} \begin{pmatrix} {}_iZ_iZ \\ {}_rZ_rZ \end{pmatrix} \right|^2 + 4({}_iY_iY)({}_iZ_iZ)[({}_rY_rZ) - ({}_iY_iZ)]^2 \leq 0.$$

Therefore the roots cannot be complex. The discriminant may be considered as the simultaneous invariant of the form

$$\begin{aligned} (a_iY + b_iZ, a_iY + b_iZ) &= 0, \\ (a_rY + b_rZ, a_rY + b_rZ) &= 0, \end{aligned}$$

for $a : b$. When our crosses have real coordinates, the roots of these two quadratic equations are conjugate complex pairs. The two quadratics can have no common roots unless they are identical. Hence the simultaneous invariant is not zero. Thus the crosses are distinct ones.

Let us take the crosses corresponding to them as $(0, 1, 0)$ $(0, 1, 0)$, $(0, 0, 1)$ $(0, 0, 1)$. Then we may express our chain in the simple form

$$\begin{aligned} {}_iX_1 &= {}_rX_1 = 0, \\ {}_iX_2 &= {}_p{}_rX_2, \\ {}_iX_3 &= {}_q{}_rX_3. \quad (p, q : \text{constant.}) \end{aligned}$$

To find the equation of surface generated, let $(\mathfrak{X})$ be the co-ordinates of a circles of our system, then

$$\begin{aligned} \mathfrak{X}_{01} &= \mathfrak{X}_{23} = 0, \\ q(\mathfrak{X}_{02} - \mathfrak{X}_{31})(\mathfrak{X}_{03} + \mathfrak{X}_{12}) &= p(\mathfrak{X}_{02} + \mathfrak{X}_{31})(\mathfrak{X}_{03} - \mathfrak{X}_{12}). \end{aligned}$$

A point (x) will lie on this circle if

$$\begin{aligned} (bx) &= 0, \quad (ax) = \cos r \sqrt{(xx)}, \\ (aa) &= 1, \quad (xx) = 1. \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathfrak{X}_{02} &= \rho x_3 \tan \varphi, \\ \mathfrak{X}_{03} &= -\rho x_2 \tan \varphi, \\ \mathfrak{X}_{31} &= \rho \frac{x_2}{x_1} \left(\sqrt{(xx)} - \frac{x_0}{\cos \varphi} \right), \\ \mathfrak{X}_{12} &= \rho \frac{x_3}{x_1} \left(\sqrt{(xx)} - \frac{x_0}{\cos \varphi} \right). \end{aligned}$$

So the required surface is given by

$$\begin{aligned} & q \left[x_1 x_3 \tan \varphi - x_2 \left(\sqrt{(xx)} - \frac{x_0}{\cos \varphi} \right) \right] \\ & \quad \left[-x_1 x_2 \tan \varphi + x_3 \left(\sqrt{(xx)} - \frac{x_0}{\cos \varphi} \right) \right] \\ &= -p \left[x_1 x_3 \tan \varphi + x_2 \left(\sqrt{(xx)} - \frac{x_0}{\cos \varphi} \right) \right] \\ & \quad \left[x_1 x_2 \tan \varphi + x_3 \left(\sqrt{(xx)} - \frac{x_0}{\cos \varphi} \right) \right] \end{aligned}$$

Or

$$(p+q)(x_2^2+x_3^2)x_1\left(\sqrt{xx}-\frac{x_0}{\cos\varphi}\right)\tan\varphi \\ + (p-q)\left[\left(\sqrt{xx}-\frac{x_0}{\cos\varphi}\right)^2+x_1^2\tan^2\varphi\right]x_2x_3=0.$$

Suppose, next, that we have

$$\rho_i X_i = a_i Y_i + b_i Z_i, \quad \sigma_r X_i = a_r Y_i + b_r Z_i,$$

where

$${}_i Y_i = p_i Z_i, \quad |{}_r Y, {}_r Z, {}_r T| \neq 0,$$

or

$${}_r Y_i = q_r Z_i, \quad |{}_i Y, {}_i Z, {}_i T| \neq 0.$$

This is a new one-parameter family of circle crosses called a *circle-strip*, or, more exactly, a *left strip* (or a *right strip*), and all crosses of the system are paratactic.

Let $({}_i X')$ $({}_r X')$ be the common perpendicular a to pair of crosses of the left circle-strip, then

$$({}_i X' {}_i Y) = 0,$$

$$({}_r X' {}_r Z) = ({}_r X' {}_r Y) = 0.$$

Hence, we may write

$${}_i X' = \lambda_i Y' + \mu_i Z'{}_i, \quad {}_r X' = {}_r Y' = {}_r Z'{}_i.$$

We thus have a right circle-strip, and each circle-strip cuts each of the others orthogonally.

A left strip of the upper layer will be represented by a point in the left plane, and a linear range in the right plane. The right strip in the lower layer will be represented by the pencil through the point in the left plane, and the line of the range in the right.

The simplest two dimensional system of circle-crosses will be, as before, the chain congruence

$${}_i X_i = a_i Y_i + b_i Z_i + c_i T_i, \quad (i)$$

$${}_r X_i = a_r Y_i + b_r Y_i + c_r T_i$$

$$|{}_i Y, {}_i Z, {}_i T| \times |{}_r Y, {}_r Z, {}_r T| \neq 0. \quad (ii)$$

We may solve the first three equations for a , b , c and substitute in the last

$$\rho_r X_i = \sum_j a_{ij} {}_i X_j, \quad |a_{ij}| \neq 0. \quad (\text{iii})$$

The crosses intersecting orthogonally pairs of crosses of this congruence will generate a second congruence of like sort called the conjugate to the first. And its equations are

$$\sigma_i U_i = \sum_j a_{ji} {}_r U_j. \quad (\text{iv})$$

The relation between the two is *reciprocal*. If we seek a cross that belongs at once to both systems, we replace $({}_i U)$ and $({}_r U)$ in the equation (iv) by $(\rho_i X)$ and $(\rho_r X)$ respectively and substitute the value $({}_r X)$ expressed by $({}_i X)$ in the right hand side of the resulting equation, then we have

$$\tau_i X_i = \sum_j b_{ij} {}_i X_j, \quad (\text{v})$$

where

$$b_{ij} = b_{ji} = \sum_k a_{ik} a_{jk}.$$

Therefore, we have a symmetric determinant equation by eliminating $({}_i X)$ from the equation (v):

$$\begin{vmatrix} b_{11} - \tau & b_{12} & b_{13} \\ b_{21} & b_{22} - \tau & b_{23} \\ b_{31} & b_{32} & b_{33} - \tau \end{vmatrix} = 0. \quad (\text{vi})$$

But as a symmetric determinant equation has always real roots, we have three real roots for τ . The crosses corresponding to the three roots of the equation (vi) are orthogonal to one another.

We, now, give the name *general chain congruence* only to a congruence of our present type where the cubic above has distinct roots. If these three crosses be looked upon as (100) (100), (010) (010), (001) (001), then the equations to our chain congruence may be reduced to the form

$${}_i X_i = a_i {}_r X_i, \quad a_1 a_2 a_3 \neq 0, \quad i = 1, 2, 3.$$

and the equations of the reciprocal congruence will be

$${}_iU_i = \frac{1}{a_i} {}_rU_i, \quad i=1, 2, 3.$$

Hence we have the following :

The circle crosses cutting orthogonally pairs of crosses of a general chain congruence generate a second such congruence. The relation between the two is reciprocal, and they have in common three orthogonal intersecting crosses.

The condition that the two circles of our two systems should be cospherical, or that one should be cospherical with the circle in bi-involution with the other, is

$$({}_iU_iX)\left[\sqrt{\sum\left(\frac{1}{a_i^2}{}_iX_i^2\right)}\sqrt{\sum(a_i^2{}_iU_i^2)}\pm\sqrt{({}_iX_iX)}\sqrt{({}_iU_iU)}\right]=0.$$

If the first factor vanish,

$$({}_iU_iX) = ({}_rU_rX) = 0,$$

the two cut orthogonally. If the second factor vanish, every circle of a cross of the first congruence for which

$$\beta\sum\left(\frac{1}{a_i^2}{}_iX_i\right) = a({}_iX_iX)$$

is cospherical with a circle of every cross of the second for which

$$a\sum({}_iU_i) = \beta({}_iU_iU).$$

A different sort of congruence will arise in the case where

$$|{}_iY_iZ_iT|=0, \quad |{}_rY_rZ_rT|\neq 0, \quad (|{}_rY_rZ_rT|=0, \quad |{}_iY_iZ_iT|\neq 0). \quad (\text{vii})$$

In this case the coordinates $({}_iY)$, $({}_iZ)$, $({}_iT)$, $({}_rY)$, $({}_rZ)$, $({}_rT)$ of the given three crosses are linearly dependent; thus the three points correspondent to $({}_iY)$, $({}_iZ)$, $({}_iT)$ [$({}_rY)$, $({}_rZ)$, $({}_rT)$] in the left (right) representing plane are colinear, i.e.

$${}_iT_i = l_iY_i + m_iZ_i, \quad ({}_rT_i = \lambda_rY_i + \mu_rZ_i). \quad (\text{viii})$$

Therefore

$$\begin{aligned}\rho_i X_i &= (a + lc)_i Y_i + (b + mc)_i Z_i, \\ [\sigma_r X_i &= (a + \lambda c)_r Y_i + (b + \mu c)_r Z_i].\end{aligned}$$

Let us now put

$$\begin{aligned}a + lc &= p, & b + mc &= q, \\ a + \lambda c &= p', & b + \mu c &= q',\end{aligned}$$

then we have

$$\begin{aligned}\rho_i X_i &= p_i Y_i + q_i Z_i, & \rho_r X_i &= p_r Y_i + q_r Z_i - c(l_r Y_i + m_r Z_i - r T_i), \\ [\sigma_r X_i &= p'_r Y_i + q'_r Z_i, & \sigma_i X_i &= p'_i Y_i + q'_i Z_i - c(\lambda_i Y_i + \mu_i Z_i - i T_i)].\end{aligned}$$

Hence the congruence will contain ∞^1 circle-strip, whose reciprocals generate the reciprocal congruence. The canonical form will be

$$\begin{aligned}{}_i X_1 &= a_1 {}_r X_1, \\ {}_i X_2 &= a_2 {}_r X_2, \\ {}_i X_3 &= 0.\end{aligned}$$

When in addition to (vii) all the first minors of the determinant

$$|{}_i Y \quad {}_i Z \quad {}_i T| \quad (|{}_r Y \quad {}_r Z \quad {}_r T|)$$

vanishes, we have a system of paratactic circle crosses.

If, on the other hand, we have

$$|{}_i Y \quad {}_i Z \quad {}_i T| = 0, \quad |{}_r Y \quad {}_r Z \quad {}_r T| \neq 0,$$

without the vanishing of the first minors of either determinant we have ∞^2 crosses which cut a given cross orthogonally. These crosses may be represented by the equations (in the lower layer)

$$\begin{aligned}\rho_i U_i &= \lambda \begin{vmatrix} {}_i Y_j & {}_i Y_k \\ {}_i Z_j & {}_i Z_k \end{vmatrix} + \mu \begin{vmatrix} {}_i Z_j & {}_i Z_k \\ {}_i T_j & {}_i T_k \end{vmatrix} + \nu \begin{vmatrix} {}_i T_j & {}_i T_k \\ {}_i Y_j & {}_i Y_k \end{vmatrix}, \\ \rho_r U_i &= \lambda \begin{vmatrix} {}_r Y_j & {}_r Y_k \\ {}_r Z_j & {}_r Z_k \end{vmatrix} + \mu \begin{vmatrix} {}_r Z_j & {}_r Z_k \\ {}_r T_j & {}_r T_k \end{vmatrix} + \nu \begin{vmatrix} {}_r T_j & {}_r T_k \\ {}_r Y_j & {}_r Y_k \end{vmatrix}.\end{aligned}$$

And the equations of this congruence may be reduced to the canonical form

$$\begin{aligned}\rho_i X_1 &= a, & \sigma_r X_1 &= b, \\ \rho_i X_2 &= b, & \sigma_r X_2 &= c, \\ \rho_i X_3 &= 0, & \sigma_r X_3 &= 0,\end{aligned}$$

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Projective Differential Geometry of Non-Developable Surfaces, III.

By

Jôyô Kanitani.

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CHAPTER IV.

CURVES OF DARBOUX AND SEGRE.

1. The curves of Darboux and Segre play a very important role in the theory of the projective differential geometry of surfaces. Many properties of them have been investigated by G. Darboux, C. Segre, G. Fubini, E. J. Wilczynski and G. M. Green from various points of view. In this chapter, I intend to add some new properties.

Let us now suppose parameter curves to be asymptotic curves. Then, in virtue of and (19) of **chap. III**, three families of Darboux curves are given by the equations

$$\sqrt[3]{C} du + w^i \sqrt[3]{B''} dv = 0, \quad (i=1, 2, 3), \quad \dots\dots\dots (1)$$

where w is an imaginary cubic root of 1.

The three families of curves given by (1) are all distinct, if C and B'' are not equal to zero. If one of C and B'' is equal to zero, e.g. $B''=0$, (1) becomes $\sqrt[3]{C} du=0$.

Therefore we know that the *three families of Darboux curves are distinct for curved surfaces, while all of them coincide with the family of generating lines for ruled surfaces*. Also from (1) we can see at once that *no two of three families of the Darboux curves can be conjugate to each other for curved surfaces*.

Since three families of Segre curves are conjugate to three families of Darboux curves, we can say the same for the Segre curves.

In this chapter, we shall investigate the properties of the curves of Darboux and Segre for curved surfaces.

Any cojugate net may be defined by an equation of the form

$$\varphi^2 du^2 - dv^2 = 0,$$

where φ is a function of u and v .

Let us put

$$d\xi = \mu(\varphi du - dv),$$

$$d\zeta = \nu(\varphi du + dv),$$

where μ and ν are integrating factors of the equations

$$\varphi du - dv = 0$$

and

$$\varphi du + dv = 0$$

respectively.

Then we have

$$\left. \begin{aligned} du &= \frac{\nu d\xi + \mu d\zeta}{2\varphi\mu\nu}, \\ dv &= \frac{-\nu d\xi + \mu d\zeta}{2\varphi\mu\nu}. \end{aligned} \right\} \dots\dots\dots (2)$$

Denote by $D_1, D'_1, D''_1, \dots$ the normal fundamental determinants in the case where the curves $\xi = \text{const.}$ and $\zeta = \text{const.}$ are parameter curves. Then we have from (17) of **chap. I**, (24) of **chap. III** and the above equation (2),

$$D_1 = -\frac{1}{D'_1} = -\frac{\nu}{\mu}. \dots\dots\dots (3)$$

From (2) and (3), we have

$$\frac{\partial^2}{\partial \xi \partial \zeta} \log \frac{D'_1}{D''_1} = 2 \frac{\partial^2}{\partial \xi \partial \zeta} \log D_1 = 2 \frac{\partial^2}{\partial \xi \partial \zeta} \log \frac{\nu}{\mu} = 2 \frac{\partial^2}{\partial u \partial v} \log \varphi. \dots\dots\dots (4)$$

But the necessary and sufficient condition that the conjugate net given by (1) should be isothermal, is

$$\frac{\partial^2}{\partial \xi \partial \zeta} \log \frac{D_1}{D''_1} = 0.$$

Therefore, from (1) and (4) we know that if any one of three families of Darboux curves form an isothermal conjugate net with a family of curves of Segre conjugate to it, then

$$\frac{\partial^2 \log \left(\frac{C}{B''} \right)}{\partial u \partial v} = 0. \quad \dots\dots\dots (5)$$

But (5) is the necessary and sufficient condition that developable surfaces of the congruences of directrices and of canonical edges correspond to a conjugate net. Therefore, we know that *three families of Darboux curves and three families of Segre curves constitute three isothermal conjugate net, if, and only if, developables of the congruences of directrices and of canonical edges correspond to a conjugate net.*

2. Put

$$\begin{aligned} h &= \frac{1}{6} \frac{\partial}{\partial u} \log \left(\frac{C}{B''} \right), \\ k &= -\frac{1}{6} \frac{\partial}{\partial v} \log \left(\frac{C}{B''} \right). \end{aligned} \quad \dots\dots\dots (6)$$

When parameter curves are asymptotic curves and $\alpha=h$, $\beta=k$, $\gamma=0$, the system of equations (1) of **chap. III** becomes

$$\left. \begin{aligned} \frac{\partial x}{\partial u} &= \rho - hx, \\ \frac{\partial x}{\partial v} &= \sigma - kx, \\ \frac{\partial \rho}{\partial u} &= h\rho + C\sigma + \mathfrak{A}_{h,k}x, \\ \frac{\partial \rho}{\partial v} &= -\tau - k\rho + \mathfrak{A}'_{k,h}x, \\ \frac{\partial \sigma}{\partial u} &= -\tau - h\sigma - \mathfrak{A}'_{h,k}x, \\ \frac{\partial \sigma}{\partial v} &= B''\rho + k\sigma - \mathfrak{A}''_{h,k}x, \\ \frac{\partial \sigma}{\partial u} &= h\tau + (\mathfrak{A}'_{h,k} - \theta)\rho - (\mathfrak{A}_{h,k} - \mathfrak{Q}_{h,k})\sigma \end{aligned} \right\} (7)$$

$$\left. \begin{aligned} & -\left(\frac{\partial}{\partial v}\mathfrak{U}_{h,k}-\frac{\partial}{\partial u}\mathfrak{U}'_{h,k}-C\mathfrak{U}''_{h,k}+6h\mathfrak{U}'_{h,k}\right)x, \\ \frac{\partial}{\partial v}=k\tau+(\mathfrak{U}''_{h,k}-\mathfrak{U}''_{h,k})\rho-(\mathfrak{U}'_{k,k}+\theta)\sigma \\ & -\left(\frac{\partial}{\partial v}\mathfrak{U}'_{k,k}-\frac{\partial}{\partial u}\mathfrak{U}''_{h,k}+B''\mathfrak{U}_{h,k}-6k\mathfrak{U}'_{h,k}\right)x. \end{aligned} \right\}$$

Let λ be any one of the roots of the cubic equation

$$x^3=-\frac{C}{B''}.$$

Then the equation

$$\frac{dv}{du}=\lambda \quad \dots\dots\dots (8)$$

defines a family of Darboux curves and the equation

$$\frac{dv}{du}=-\lambda \quad \dots\dots\dots (9)$$

defines a family of Segre curves conjugate to the family of curves defined by (8).

Tangents of the family of curves defined by (8) constitute a congruence. On every tangent line of the congruence, there are two focal points. Let t be a tangent line of the congruence which touches at a point $P(x)$ to a curve C of the family defined by (8). One of the focal points on t is P . The other focal point is called by Wilczynski the *ray point* of P with respect to the curve C .

Any point on the line t , except the point $P(x)$, may be defined by an expression of the form

$$\phi=\rho+\lambda\sigma+\mu x.$$

For ϕ be the ray point on t with respect to the curve C , it is necessary and sufficient that four points $x, \phi, \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}$ are co-planar.

From this reason, by the same method as employed in **n° 5 chap. III**, making use of (7), we can easily see that the ray point may be defined by the expression

$$\rho+\lambda\sigma, \quad \dots\dots\dots (10)$$

Similarly, we can see that the ray point of $P(x)$ on the tangent t' at P of the curve C' of the family of curves defined by (9) with respect to C' may be defined by the expression

$$\rho - \lambda\rho + B''\lambda^2x. \dots\dots\dots(11)$$

Therefore, the ray of $P(x)$ with respect to the conjugate curves C and C' , (i.e. the line which connects the points (10) and (11)), passes through the two points

$$\rho + \frac{B''\lambda^2}{2}x, \quad \sigma - \frac{B''\lambda}{2}x.$$

The reciprocal line of this ray passes through the point x and through the point

$$\tau + \frac{B''\lambda}{2}(\rho - \lambda\sigma). \dots\dots\dots(11)$$

The osculating plane at x of the curve C is given by the equation

$$\left| \begin{array}{ccc} \xi & x & \frac{dx}{du} \frac{d^2x}{du^2} \end{array} \right| = 0, \dots\dots\dots(12)$$

where (ξ) are current coordinates, and $\frac{dx}{du}$ and $\frac{d^2x}{du^2}$ denote the differentiations along C .

But we have from (7) and (8)

$$\left. \begin{aligned} \frac{dx}{du} &= \rho + \lambda\sigma - (h + k\lambda)x, \\ \frac{d^2x}{du^2} &= -2\lambda\tau + (B''\lambda^2 - 2k\lambda)\rho + (C' - 2k\lambda^2)\sigma \\ &\quad + \{A - kC - \lambda^2(A'' + hB'') + 2k^2\lambda^2\}x. \end{aligned} \right\} (13)$$

If we substitute the values of $\frac{dx}{du}$ and $\frac{d^2x}{du^2}$ given by (13) in (12), (12) becomes

$$\left| \begin{array}{ccc} \xi & x & \rho + \sigma\lambda - \tau - \frac{B''\lambda}{2}(\rho - \sigma\lambda) \end{array} \right| = 0. \dots\dots\dots(14)$$

Similarly, we can see that the osculating plane at x of the curve C' is given by the equation

$$| \xi \ x \ \rho - \lambda \ \tau | = 0, \dots\dots\dots(15)$$

Therefore, the axis of x with respect to the conjugate curves C and C' , (i.e. the line of the intersection of the planes given by (14) and (15)), passes through the point x and through the point

$$\tau - \frac{B''\lambda}{2}\rho + \frac{B''\lambda^2}{2}\sigma.$$

The reciprocal line of this axis connects the points

$$\rho - \frac{B''\lambda^2}{2}x, \quad \sigma + \frac{B''\lambda}{2}x.$$

Summing up the above results, we have the following two theorems which are dual to each other. (Fig. 1).

Let P be an ordinary point on a curved surface. Three Darboux curves and three Segre curves which meet at P constitute three pairs of conjugate curves. Corresponding to these three pairs of conjugate curves, there are three rays and three axes of P .

The Reciprocal line of any one of these axes passes through the corresponding ray point on the tangent at P of the curve of Darboux passing through P .

Three ray points of P with respect to the three curves of Darboux passing through P lie on the line which is the harmonic conjugate of the tangent at P of any one of these curves of Darboux with respect to the corresponding ray and the reciprocal line of the axis of P .

The Reciprocal line of any one of these rays lies on the corresponding osculating plane at P of the curve of Segre passing through P .

Three osculating planes at P of the three curves of Segre passing through P intersect at the line which is the harmonic conjugate of the tangent at P of any one of these curves of Segre with respect to the corresponding axis and the reciprocal line of the ray of P .

3. Let $P(x)$ be a point on the surface. Two asymptotic tangents at P and three rays of P with respect to the three conjugate curves formed by curves of Darboux and Segre passing through P determine a conic K which touches these five lines. We shall now find the equation of K referring to the moving coordinate frame of reference of

which the vertices are at the points x, ρ, σ , and the unit point is at the point $x+\rho+\sigma$.

Line coordinates of the said rays referring to the moving coordinate of reference are

$$\left. \begin{aligned} (1, -\tfrac{1}{2}B''\lambda^2, \tfrac{1}{2}B''\lambda), \\ (1, -\tfrac{1}{2}w^2B''\lambda^2, \tfrac{1}{2}wB''\lambda) \\ \text{and} \quad (1, -\tfrac{1}{2}wB''\lambda^2, \tfrac{1}{2}w^2B''\lambda) \end{aligned} \right\} \dots\dots\dots (16)$$

respectively, where w is an imaginary cubic root of 1.

Let ν_i ($i = 0, 1, 2$) be line coordinates referring to the moving coordinate frame of reference. The equation in line coordinates of a conic which touches the line $x\rho$ and $x\sigma$ is of the form

$$\nu_0(a_0\nu_0 + a_1\nu_1 + a_2\nu_2) + a_3\nu_1\nu_2 = 0. \dots\dots\dots (17)$$

If the conic given by (17) touch the three lines, the line coordinates of which are given by (16), the coefficients a_0, a_1, a_2, a_3 must satisfy the equations

$$\left. \begin{aligned} B''a_1\lambda^2 - B''a_2\lambda - 2\left(a_0 + \frac{\theta}{4}a_3\right) &= 0, \\ w^2B''a_1\lambda^2 - wB''a_2\lambda - 2\left(a_0 + \frac{\theta}{4}a_3\right) &= 0, \\ wB''a_1\lambda^2 - w^2B''a_2\lambda - 2\left(a_0 + \frac{\theta}{4}a_3\right) &= 0. \end{aligned} \right\} \dots\dots\dots (18)$$

From (18) we have

$$\begin{aligned} a_1 &= a_2 = 0, \\ \frac{a_0}{a_3} &= -\frac{\theta}{4}. \end{aligned}$$

Therefore, the equation in line coordinates of K is

$$\theta\nu_0^2 = 4\nu_1\nu_2. \dots\dots\dots (19)$$

Line coordinates of three reciprocal lines of three axes of P with respect to three pairs of the conjugate curves formed by curves of Darboux and Segre passing through P are

$$\begin{aligned} (1, \tfrac{1}{2}B''\lambda^2, -\tfrac{1}{2}B''\lambda), \\ (1, \tfrac{1}{2}w^2B''\lambda^2, -\tfrac{1}{2}B''w\lambda), \\ (1, \tfrac{1}{2}wB''\lambda^2, -\tfrac{1}{2}w^2B''\lambda), \end{aligned}$$

respectively. Therefore, these three lines touch the conic K .

Let w_i ($i = 0, 1, 2$) be point coordinates referring to the moving coordinate frame of reference. Then, from (19) we see that the equation of K in point coordinates is

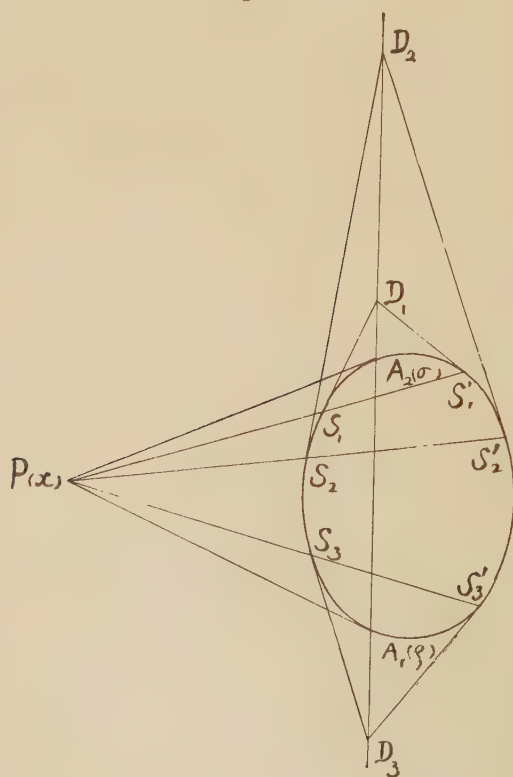
$$w_0^2 = \theta w_1 w_2. \quad \dots\dots\dots (20)$$

From (19) or (20), we can see at once that the lines $x\rho$ and $x\sigma$ touch K at the points ρ and σ respectively. Therefore, the line $\rho\sigma$ is the polar line of the point x with respect to K . But three ray points D_1, D_2, D_3 of P with respect to the three curves of Darboux passing through P lie on the line $\rho\sigma$. Therefore, three polar lines of D_1, D_2 and D_3 pass through P and they are tangents at P of three Segre curves passing through P .

Therefore, we have the following theorem.

Let P be an ordinary point on a curved surface and D_1, D_2, D_3 be three ray points of P with respect to the three curves of Darboux

Fig. 1.



passing through P .

Then the points D_1, D_2, D_3 lie on a line which intersects the asymptotic tangents at in the points A_1 and A_2 . Let S_1, S_2, S_3 be three ray points of the three curves of Segre passing through P , PS_1, PS_2, PS_3 being tangents conjugate to the tangents PD_1, PD_2, PD_3 respectively. Let $D_1S'_1, D_2S'_2, D_3S'_3$ the reciprocal lines of axes of P with respect to three pairs of the conjugate curves formed by the curve of Darboux and Segre passing through P , S'_1, S'_2, S'_3 being the

points on the tangents PS_1, PS_2, PS_3 respectively (Fig. 1).

Then, there is a conic which touches eight lines $PA_1, PA_2; D_1S_1, D_1S'_1; D_2S_2, D_2S'_2; D_3S_3, D_3S'_3$ and $A_1, A_2; S_1S'_1; S_2S'_2; S_3S'_3$ are the points of the contact.

We shall call this conic the *canonical ray conic* of P .

Similarly, we can prove the following theorem.

Let p be a tangent plane at an ordinary point P of a curved surface. Let s_1, s_2, s_3 be three osculating planes at P of the three curves of Segre passing through P . Then s_1, s_2, s_3 intersect at a line which determines planes a_1, a_2 with asymptotic tangents at P . Let d_1, d_2, d_3 be three osculating planes at P of the three curves of Darboux passing through P , pd_1, pd_2, pd_3 being tangents conjugate to the tangents ps_1, ps_2, ps_3 respectively. Let $d'_1s_1, d'_2s_2, d'_3s_3$ be the reciprocal lines of three rays of P with respect to three conjugate curves formed by the curves of Darboux and Segre passing through P , d'_1, d'_2, d'_3 being planes containing the tangents pd_1, pd_2, pd_3 respectively,

Then, there is a cone of the second degree which passes through eight lines $pa_1, pa_2; d_1s_1, d'_1s_1; d_2s_2, d'_2s_2; d_3s_3, d'_3s_3$ and the planes $a_1, a_2, d_1, d'_1; d_2, d'_2; d_3, d'_3$ are tangent planes of the cone.

We shall call this cone the *canonical axis cone* of P .

In n°8 chap. III, we proved that, if

$$\alpha = l \frac{|\mathfrak{L} \mathfrak{C} D|}{\theta} + \frac{1}{2} \frac{\theta_u}{\theta},$$

$$\beta = l \frac{|\mathfrak{L} \mathfrak{B} D|}{\theta} + \frac{1}{2} \frac{\theta_v}{\theta},$$

l being a constant, the lines $\rho\sigma$ and $\alpha\tau$ are invariant lines. But when the parameter curves are asymptotic curves,

$$-\frac{1}{3} \frac{|\mathfrak{L} \mathfrak{C} D|}{\theta} + \frac{1}{2} \frac{\theta_n}{\theta} = \frac{1}{6} \frac{\partial}{\partial u} \log \left(\frac{C}{B''} \right),$$

$$-\frac{1}{3} \frac{|\mathfrak{L} \mathfrak{B} D|}{\theta} + \frac{1}{2} \frac{\theta_v}{\theta} = -\frac{1}{6} \frac{\partial}{\partial v} \log \left(\frac{C}{B''} \right).$$

Therefore we know that, referring to any system of parameter

curves, if

$$\alpha = -\frac{1}{3} \frac{|\mathfrak{L} \mathfrak{C} D|}{\theta} + \frac{1}{2} \frac{\theta_u}{\theta},$$

$$\beta = -\frac{1}{3} \frac{|\mathfrak{L} \mathfrak{B} D|}{\theta} + \frac{1}{2} \frac{\theta_v}{\theta},$$

then $\rho\sigma$ is the line on which the three ray points of x with respect to three Darboux curves passing through x lie and $x\tau$ is the line at which intersect the three osculating planes at x of the three curves of Segre passing through x .

4. To any point of the surface, we can associate a canonical ray conic. Therefore, these conics constitute a congruence. On each conic, there are six focal points¹. We shall determine these focal points.

Let (w_0, w_1, w_2) be projective coordinates of a point Q on the conic K which lie on the tangent plane at $P(x)$ on the surface, referring to the moving coordinate frame of reference.

Put

$$\frac{w_1}{w_0} = t.$$

Then we have from (20)

$$w_0 : w_1 : w_2 = 1 : t : \frac{1}{\theta t}.$$

Projective coordinates (ξ) of Q referring to the space fixed coordinate frame of reference may be expressed in the form

$$\xi = x + t\rho + \frac{1}{\theta t}\sigma, \dots\dots\dots (21)$$

if the proportional factor properly chosen, for the unit point of the moving coordinate frame of reference is $x + \rho + \sigma$.

If Q be a focal point on K , then, since all the surfaces of the congruence which passes through K must have the tangent plane common at Q , four points $\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial u}, \frac{\partial \xi}{\partial v}$ should be coplanar.

¹ Darboux, Leçons sur la théorie générale des surfaces. Vol. II. p. 5.

From this reason, by the usual method, we can see that for Q to be a focal point, it is necessary and sufficient that

$$C(\theta t^2)^3 - B'' - 2t\{(\theta t^2)^2\mathfrak{X}_{h,k} - 2\mathfrak{X}'_{h,k}(\theta t^2) + \mathfrak{X}''_{h,k}\} \\ + \left(2h - \frac{\theta_u}{\theta}\right)(\theta t^2)^2 - \left(2k - \frac{\theta_v}{\theta}\right)(\theta t^2) = 0 \dots \dots \dots (22)$$

Six roots of this equation determine the six focal points on K .

Now let us suppose that six focal points of any canonical ray conic lie on three lines passing through the point at which the plane of the conic touches the surface so that on each one of the three lines lie two focal points.

Then, six roots of (22) must be of the form

$$\pm t_1, \pm t_2, \pm t_3.$$

Accordingly,

$$(\theta t_i^2)^2\mathfrak{X}_{h,k} - 2\mathfrak{X}'_{h,k}(\theta t_i)^2 + \mathfrak{X}''_{h,k} = 0, \quad (i=1, 2, 3) \dots \dots (23)$$

From (23) we have

$$\mathfrak{X}_{h,k} = \mathfrak{X}''_{h,k} = 0, \dots \dots \dots (24)$$

$$\mathfrak{X}'_{h,k} = 0. \dots \dots \dots (25)$$

From (6) and (25) we have

$$\frac{\partial}{\partial u \partial v} \log \frac{C}{B''} = 0.$$

Therefore,

$$\frac{C}{B''} = \frac{U(u)}{V(v)}, \dots \dots \dots (26)$$

where $U(u)$ and $V(v)$ are functions of u only and of v only respectively.

Carry out the transformation

$$\tilde{u} = \int \sqrt[3]{U} du, \quad \tilde{v} = \int \sqrt[3]{V} dv. \dots \dots \dots (27)$$

Then, in virtue of (20) in chap. I, we have

$$\frac{\bar{C}}{\bar{B}''} = 1.$$

Let us assume that this transformation has been made.

Then,

$$C = B'' \quad h = k = 0, \dots \dots \dots (28)$$

and (24) becomes

$$A=A''=0. \dots\dots\dots(29)$$

When the parameter curves are asymptotic curves, the system of the equations (4) in chap. II¹ becomes

$$\left. \begin{aligned} 2\frac{\partial A}{\partial v} + \frac{\partial^2 C}{\partial v^2} &= \frac{\partial \theta}{\partial u} + C\frac{\partial B''}{\partial u}, \\ -2\frac{\partial A''}{\partial u} + \frac{\partial^2 B''}{\partial u^2} &= \frac{\partial \theta}{\partial v} + B''\frac{\partial C}{\partial v}, \\ \frac{\partial^2 A}{\partial v^2} + \frac{\partial^2 A''}{\partial u^2} - \frac{\partial(A''C)}{\partial v} - \frac{\partial(AB'')}{\partial u} - A''\frac{\partial C}{\partial v} - A\frac{\partial B''}{\partial v} &= 0. \end{aligned} \right\} (30)$$

From (28), (29), and (30) we have

$$\left. \begin{aligned} \frac{3}{2}\frac{\partial \theta}{\partial u} &= \frac{\partial^2 C}{\partial v^2}, \\ \frac{3}{2}\frac{\partial \theta}{\partial v} &= \frac{\partial^2 C}{\partial u^2}. \end{aligned} \right\} \dots\dots\dots(31)$$

From (7), (28) and (29) we have the system of equations

$$\left. \begin{aligned} \frac{\partial \rho}{\partial u} &= C\sigma, \\ \frac{\partial \rho}{\partial v} - \frac{\partial \sigma}{\partial u} &= -\tau, \\ \frac{\partial \sigma}{\partial v} &= C\rho, \\ \frac{\partial \tau}{\partial u} &= -\theta\rho - C_v\sigma, \\ \frac{\partial \tau}{\partial v} &= -C_u\rho - \theta\sigma, \end{aligned} \right\} \dots\dots\dots(32)$$

which is completely integrable in virtue of (31).

Let (r_i, s_i, t_i) ($i=1, 2, 3$) be a system of solutions such that

$$\begin{vmatrix} r & s & t \end{vmatrix} \neq 0.$$

¹ J. Kanitani, Mem. Coll. Sci. Kyoto, **6**, p. 214. These equations are misprinted in that page. The left-hand sides of the first and the second of them should be respectively equal to $f^{(1)}$ and $f^{(2)}$ given in p. 210 and p. 211 of the Mem. cited.

Then,

$$\begin{aligned}\rho_i &= \sum_{j=1}^3 C_{ij} r_j, \\ \sigma_i &= \sum_{j=1}^3 C_{ij} s_j, \\ \tau_i &= \sum_{j=1}^3 C_{ij} t_j,\end{aligned}\quad (i=1, 2, 3, 4),$$

Therefore, the points ρ and σ lie on the plane

$$\begin{vmatrix} \eta_1 & C_{11} & C_{12} & C_{13} \\ \eta_2 & C_{21} & C_{22} & C_{23} \\ \eta_3 & C_{31} & C_{32} & C_{33} \\ \eta_4 & C_{41} & C_{42} & C_{43} \end{vmatrix} = 0,$$

i.e. the line $\rho\sigma$ on which three ray point of the point x with respect to the curves of Darboux passing through x , describes a plane, when x moves on the surface.

Next, let us assume that the line $\rho\sigma$ describes a plane when x moves on the surface. Then the points $\rho, \sigma, \frac{\partial\rho}{\partial u}, \frac{\partial\sigma}{\partial v}, \frac{\partial\sigma}{\partial u}, \frac{\partial\rho}{\partial v}$ should lie in a plane. Accordingly, in virtue of (7),

$$\mathfrak{A}_{h,k} = \mathfrak{A}'_{h,k} = \mathfrak{A}''_{h,k} = 0.$$

Therefore, we know that *six focal points on a canonical ray conic K lie on three lines passing through the point P at which the plane of K touches the surface, so that on any one of the three lines lie two focal points, if, and only if, the line on which the three ray points of P with respect to the curves of Darboux passing through P lie describes a plane, when P moves on the surface.*

5. To any point of the surface, we can associate a canonical axis cone. Therefore, to the totality of the points of the surface, correspond an assemblage of ∞^2 axis cones which is the space dual of the congruence of conics. We shall call also this assemblage the *congruence* of the axis cones. Tangent planes of the manifoldness of ∞^2 of ∞^1 canonical axis cones corresponding to ∞^1 points of a curve L on the

surface determine a surface which is the envelope of these ∞^2 tangent planes and which we shall call the surface of the congruence corresponding to the curve L .

Among ∞^1 tangent planes of a canonical axis cone C , there are six focal planes any one of which touches all the surfaces of the congruence circumscribed by C at the same point. We shall now find these six focal planes of C .

In the same manner as in n°4, we can see that, referring to the moving coordinate frame of reference of which the vertices are at the points x, ρ, σ, τ and the unit point is at the point $x+\rho+\sigma+\tau$, the equation in point coordinates of C is

$$\theta w_3^2 = 4w_1w_2. \quad \dots\dots\dots (33)$$

Put

$$\frac{w_1}{w_3} = t.$$

Then, for the points on C , in virtue of (33)

$$w_1 : w_2 : w_3 = t : \frac{\theta}{4t} : 1.$$

Projective coordinates (η) of a point Q on a canonical axis cone corresponding to a point P on the surface, referring to the space fixed coordinates frame of reference, may be expressed in the form

$$\eta = \tau + t\rho + \frac{\theta}{4t}\sigma + \nu x,$$

if the proportional factor be properly chosen.

If Q be a point on the curve along which the cone C circumscribes the surface of the congruence corresponding to the curve on the surface which is defined by the equation

$$\varphi(u, v) = 0,$$

then four points $x, \eta, \frac{\partial \eta}{\partial t}, \frac{\partial \eta}{\partial u} du + \frac{\partial \eta}{\partial v} dv$ should be coplanar.

From this reason, by the usual method, we can see that

$$\left[Ct^3 - \left(\mathfrak{U}_{h,k} - \mathfrak{L}_{h,k} \right) t^2 + \left(\mathfrak{U}'_{h,k} - \theta \right) \frac{\theta}{4} \right. \\ \left. + \left(\frac{\theta_u}{\theta} - 2h \right) \frac{\theta t}{4} + \left(\nu + \frac{\theta}{2} \right) \frac{\theta}{4} \right] du \\ + \left[\frac{B'' \theta^2}{16} - \left(\mathfrak{U}'_{h,k} + \theta \right) t^3 + \left(\mathfrak{U}''_{h,k} - \mathfrak{L}''_{h,k} \right) \frac{\theta t}{4} \right. \\ \left. + \left(\frac{\theta_v}{\theta} - 2k \right) \frac{t^2}{4} + \left(\nu + \frac{\theta}{2} \right) t^3 \right] dv = 0 \quad \dots (31)$$

If Q be the point of the contact of one of the focal planes, the value of ν given by (31) should be independent of $du : dv$. Therefore, in this case, t must be a root of the equation.

$$Ct^6 - B'' \left(\frac{\theta}{4} \right)^3 - t \left\{ \left(\mathfrak{U}_{h,k} - \mathfrak{L}_{h,k} \right) t^4 - 2 \mathfrak{U}'_{h,k} \frac{\theta t^2}{4} + \left(\mathfrak{U}''_{h,k} - \mathfrak{L}''_{h,k} \right) \left(\frac{\theta}{4} \right)^2 \right\} \\ + \frac{\theta t^2}{4} \left\{ \left(\frac{\theta_u}{\theta} - 2h \right) t^2 - \left(\frac{\theta_v}{\theta} - 2k \right) \frac{\theta}{4} \right\} = 0 \dots \dots (32)$$

Six roots of this equation determine the six focal plane of C .

Now let us suppose that six focal planes of any canonical axis cone intersect at three lines with the tangent plane of the surface at the vertex of the cone so that at each one of these lines, intersect two focal planes. Then, in the same manner as in **n°5**, we can see that

$$\mathfrak{U}_{h,k} - \mathfrak{L}_{h,k} = \mathfrak{U}''_{h,k} - \mathfrak{L}''_{h,k} = 0, \quad \dots \dots \dots (33)$$

$$\mathfrak{U}'_{h,k} = 0.$$

Therefore, also in this case, $\frac{C}{B''}$ must be of the form of (26).

Suppose that the transformation of the form of (27) has been made.

Then,

$$C = B'', \quad h = k = 0. \quad \dots \dots \dots (34)$$

From (33) and (34) we have

$$A + \frac{\partial C}{\partial v} = 0, \quad A'' - \frac{\partial C}{\partial u} = 0. \quad \dots \dots (35)$$

From (30), (34) and (35) we have

$$\left. \begin{aligned} \frac{3}{2} \frac{\partial \theta}{\partial u} &= -\frac{\partial^2 C}{\partial v^2}, \\ \frac{3}{2} \frac{\partial \theta}{\partial v} &= -\frac{\partial^2 C}{\partial u^2}. \end{aligned} \right\} \dots\dots\dots (36)$$

From (7), (34), (35) and (36) we have

$$\frac{\partial}{\partial n}(\tau + \theta x) = 0, \quad \frac{\partial}{\partial v}(\tau + \theta x) = 0.$$

Therefore, the lines $x\tau$ constitute a bundle of lines. Reciprocally, if the lines $x\tau$ constitute a bundle of lines, making use of (7), we can easily see that

$$\mathfrak{A}_{h,k} - \mathfrak{Q}_{h,k} = \mathfrak{A}'_{h,k} = \mathfrak{A}''_{h,k} - \mathfrak{Q}''_{h,k} = 0.$$

Therefore, we know that *six focal planes of any canonical axis cone intersect at three lines with the tangent plane of the surface at the vertex of the cone so that at each one of these lines intersect two focal planes, if and only if, the lines at which intersect three osculating planes of three Segre curves which meet at a point on the surface, constitute a bundle of lines.*

On the Spontaneous Decomposition of Lecithin.

By

Shichizo Kato and Osamu Shinoda.

(Received Dec. 12, 1923)

It is well known that to get lecithin in a pure state, special care must be taken in the preparation from animal or plant materials, employing pure reagents and protecting it from oxidation, and also in its preservation, avoiding the action of air, light and moisture as far as possible.¹

When lecithin is treated with mineral acids², it yields choline. According to E. Fourneau and Piettre³, however, unsaturated fatty acids in lecithin were removed by alcoholysis.

The present research was, therefore, undertaken to see what chemical changes would occur when lecithin was kept in a reagent bottle at ordinary temperature for a long time.

The material used for this experiment consisted of two samples, one a dark plastic mass containing a white crystalline substance, and the other with the same appearance but lacking the crystalline substance, and both were marked "extra pure lecithin from eggs, made by E. Merck, Germany."

The former sample was secured through the courtesy of Prof. I. Matsuo, of our Medical College, and was bought in Germany eight years ago, and kept in his laboratory until 1921; and the latter was bought in 1922, in Kyoto.

The first sample, on analysis, after drying to constant weight in vacuo, gave the following results:

¹ H. Maclean: *Lecithin and Allied Substances*, 35-36 (1918).

² G. Maruzzi: *Zs. Physiol. Chem.*, **55**, 352 (1908).

³ *Bull. Soc. Chim.*, (4); **11**, 805 (1912).

0.1355 grm. of the substance gave 0.2918 grm. CO₂ and 0.1409 grm. H₂O.
 0.1313 grm. of the substance gave 0.0037 grm. N₂ (Kjeldahl's method).
 0.1700 grm. of the substance gave 0.0219 grm. Mg₂P₂O₇.

C=59.61; H=11.80; N=7.82; P=3.58.

$$\frac{\text{NH}_2\text{-N}}{\text{total N}} = \frac{57}{100}; \text{N:P} = 1:0.6.$$

As to the lecithin from Merck, J. Bang has already stated in the following words¹; "Es enthielt 2.05% N und 3.35% P. Die richtige Formel ist C₄₃H₈₀NPO₉ mit 1.35% N und 3.9% P. Mercks Preparat ist demnach sehr unrein und zu wissenschaftlichen Zwecken vollkommen unbrauchbar."

The so-called pure lecithin from Merck analysed by the authors, seems impure when our analytical figures compared with those of the lecithin prepared by Stern and Thierfelder² from egg yolk, under special precautions:

C=64.63; H=10.96; N=2.08; P=3.97; P:N=1:1.16.

Our second sample, however, showing the following analytical results, strongly resembles in its elementary composition Stern and Thierfelder's sample:

0.0909 grm. of the substance gave 0.2129 grm. CO₂ and 0.0879 grm. H₂O.
 0.1420 grm. of the substance gave 0.0182 grm. Mg₂P₂O₇.
 0.0949 grm. of the substance gave 0.0022 grm. N₂ (by Kjeldahl's method).

C=64.00; H=10.74; N=2.32; P=3.59; N:P=1.1; 1;

$$\frac{\text{NH}_2\text{-N}}{\text{total N}} = \frac{32}{100}.$$

20 grm. of the first sample, dissolved in ether and concentrated to a small volume, and poured into dry acetone to precipitate the lecithin. The brown-colored precipitate, separated from the solution, dried under reduced pressure and redissolved in ether. The precipitation of the lecithin from the ether solution with acetone, was repeated three times and the final product was analysed after drying in vacuo. It amounted to 8 grm.

0.1951 grm. of the substance gave 0.3639 grm. CO₂ and 0.1800 grm. H₂O.
 0.1276 grm. of the substance gave 0.0242 grm. Mg₂P₂O₇.
 0.1044 grm. of the substance gave 0.0016 grm. N₂ (Kjeldahl's method).

¹ Chem. u. Biochem. d. Lipoide, 50 (1911).

² Z. Physiol. Chem., 53, 379 (1907).

C=50.88 ; H=10.35 ; N=1.53 ; P=5.28 ; N : P=1 : 1.4 ;

$$\frac{\text{NH}_2\text{-N}}{\text{total N}} = \frac{39}{100}.$$

The purified sample, as seen by the foregoing experimental results, contains less carbon and more phosphorus than the true lecithin, and this discrepancy between fact and theory indicates that some of the fatty acids have been removed from the lecithin molecule.

In order to discover the nature of the fatty acids remaining in the glycerin-phosphoric fragment of lecithin, 7 grm. of the purified substance was hydrolysed with 10% alcoholic soda solution, and the sodium salts isolated from the reaction products were converted into lead salts which are insoluble in ether.

The free acids obtained from the lead salt, assumed to consist of stearic and palmitic acids, were induced to separate from each other by means of the fractional crystallization of their magnesium salts in 98% alcohol solution according to the direction of Heinz¹.

It did not, however, yield the desired result, the melting point of the nine fractions ranging from 52 to 57°.

The acetone solution separated from the insoluble lecithin-like compound described above, was concentrated under reduced pressure to a thick syrup, amounting to 3 grm., and dissolved in absolute ether. On passing dry ammonia gas in the ethereal solution, a white precipitate formed separate from the solution, and one part of the ammonium salt was converted into calcium salt, and the remainder into lead salt, and both salt were analysed :

0.124 grm. of the calcium salt gave 0.2887 grm. CO₂, 0.1252 grm. H₂O and 0.0041 grm. ash.

C=63.37 ; H=11.27 ; Ash=2.45.

The free acid obtained from the calcium salt, melting at 52-53°. 0.1027 grm. of the substance gave 0.2718 grm. CO₂ and 0.1190 grm. H₂O,

C=72.19 ; H=12.96.

¹ J. prak. Chem., 66, 12 (1855).

The solution of 0.1021 grm. requires 3.9 c.c. N/10 NaOH solution for neutralization,

whence M.W.=270.

The other sample of the free acid prepared from the lead salt, gave on analysis the following results :

0.1032 grm. substance gave 0.2767 grm. CO₂ and 0.1188 grm. H₂O.

C=73.13 ; H=12.88 ; M.W.=272 ; M.P. 52-53.°

To our surprise, the acids isolated as a cleavage product of lecithin so far as we had studied their elementary compositions, molecular weights, melting points, and chemical properties, were also found to consist of a mixture of stearic and palmitic acids.

The second sample purified by fractional precipitation seven times as usual, was found to contain the following composition by analysis :

0.1129 grm. substance gave 0.1009 grm. H₂O, 0.2616 grm. CO₂ and 0.0099 grm. ash.

0.125 grm. gave ammonia by Kjeldahl, method which requires 1.91 c.c. N/10 HCl to neutralise.

0.4851 grm. substance gave 0.1194 grm. Agl.

C=63.25 ; H=9.94 ; Ash=8.75 ; N=2.14 ;

$$\frac{\text{NH}_2\text{-N}}{\text{total N}} = \frac{16}{100} ; \quad \text{Glycerin} = 9.66.$$

As seen in the foregoing results, lecithin, on keeping, suffers spontaneous decomposition and the fatty acids were gradually separated from the glycerin phosphoric acid part, and the products resulting were poor in carbon and hydrogen but rich in phosphorus.

The acetone solution supposed to contain the fatty acids which broke off from the lecithin molecule, was concentrated to a syrup under reduced pressure, the fatty acids were extracted with ether, and dry ammonia gas passed in the ethereal solution, whereby the ammonium salt was precipitated which was then transformed into lead salt as usual.

The lead salts, thus obtained, differing from those of the oleic and linoleic acids in their solubility in ether, were assumed to be saturated compounds.

The molecular weight of the mixed acids showed 265 which corresponds to the mean value of that for stearic and palmitic acids.

The mixture was subjected in 98% alcohol solution to a fractional precipitation with acetate of magnesium. After repeating the precipitation 17 times, it was divided into 7 portions, and each portion washed with alcohol and ether and then decomposed with hydrochloric acid. The free acid of each fraction was examined as to its melting point and molecular weight and the results were compared with those of Heinz¹ in his investigation of the mixture of stearic and palmitic acids.

Fraction	gram.	M.P.	M.W.
1.	0.1	63—64°	
2.	0.15	56°	282.
3.	0.15	54°	
4.	0.2	semisolid	
5.	0.4	61°	
6.	0.2	57—58°	272.
7.	0.35	semisolid	

According to Heinz's table, our less soluble fraction melting at 63–64°, will be considered as a mixture of 70% stearic acid and 30% palmitic acid, on the contrary, the more soluble fraction melting at 61° predominates in palmitic acid (95%).

Summary.

1. Pure ovo-lecithin from Merck was analysed, and was confirmed to be a mixture of true lecithin and cephalin.

2. While ovo-lecithin keeps at ordinary temperature for a long time, the fatty acids were gradually separated from the glycerine-phosphoric acid part, and a product resulted which was rich in phosphorus and nitrogen but poor in carbon and hydrogen.

Aug. 1923. Laboratory of Organic-and Bio-chemistry.

¹ Loc. cit.

A Theorem Relating to the Orders of a Transcendental Integral Function of Two Independent Variables.

By

Ryô Yasuda.

(Received February 19, 1924)

Definition. Let $f(z, z')$ be a transcendental integral function of z for any z' in a finite domain $D(z')$, and be a regular function of z' in $D(z')$ for any finite z .

If ρ be a finite number such that

i) for any value of z' in $D(z')$ and for any prescribed positive value ϵ , there corresponds a positive value $R \equiv R(\epsilon, z')$ such that

$$\left| f(z, z') \right| \leq e^{r^{\rho + \epsilon}} \quad \text{for all } |z| = r \geq R,$$

ii) for a certain value z' in $D(z')$ and for any prescribed positive value ϵ ,

$$\left| f(z, z') \right| > e^{r^{\rho - \epsilon}} \quad \text{for infinitely many values of } |z| = r \text{ which}$$

diverges without limit.

ρ is called the *order* of $f(z, z')$ in z for z' in $D(z')$.

Similarly we may define the order of $f(z, z')$ in z' .

Definition. Let $E(z, z')$ be a field such that z' is any point in a finite domain $D(z')$ while z is any finite point in the whole Gauss plane, and $f(z, z')$ be a regular function of z and z' in $E(z, z')$. If $f(z, z')$ be a transcendental integral function of order ρ in z such that for any prescribed positive value ϵ and for all z' in $D(z')$, there cor-

responds a positive value R (independent of z') and we have

$$\left| f(z, z') \right| \leq e^{r^{\rho+\varepsilon}} \quad \text{for all } |z|=r \geq R,$$

then $f(z, z')$ is called a *uniformly increasing function* of the ρ^{th} order in z for all z' in $D(z')$.

Note. If $f(z, z')$ be a uniformly increasing function of the ρ^{th} order in z for all values of z' in the vicinity of any point in $D(z')$, it will also be true, by an extension of the Heine-Borel theorem¹, for all z' in $D(z')$.

Theorem. Let $f(z, z')$ be a transcendental integral function of z and z' and be a uniformly increasing function of the ρ^{th} order in z for all z' in a finite domain $D(z')$. Let $S = \{a_1, a_2, a_3, \dots\}$ be a set of points in the z -plane. If $f(a_i, z') \equiv P_i(z')$, ($i=1, 2, 3, \dots$), where $P_i(z')$ is a polynomial of the n_i^{th} degree ($n_i < N$: a finite number), then there will be no limiting point of S at finiteness and the exponent of convergence of $|a_i|$, ($i=1, 2, 3, \dots$), will not be greater than ρ .

$$\text{As } f(a_i, z') \equiv P_i(z'), \quad (i=1, 2, 3, \dots), \quad \frac{\partial^N f(a_i z')}{\partial z'^N} \equiv 0 \text{ for } i=1,$$

$2, 3, \dots$ Accordingly S has no limiting point at finiteness, unless $f(z, z')$ should be a mere polynomial of z' for any value of z , which is contrary to the assumption. Thus the first part of the theorem is proved.

As $f(z, z')$ is a uniformly increasing function of the ρ^{th} order in z for all z' in $D(z')$, we have

$$\left| \frac{\partial f(z, z')}{\partial z'} \right| = \left| \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f(z, \zeta)}{(\zeta - z')^2} d\zeta \right| \leq \frac{e^{r^{\rho+\varepsilon}}}{r'} \leq e^{r^{\rho+\varepsilon}} \quad \text{for all } |z|=r \geq R,$$

where C is a circle contained in $D(z')$, centre the point z' and radius equal to r' . For a fixed value of r' , ε' may be taken as small as we please. Accordingly $\frac{\partial f(z, z')}{\partial z'}$ is a transcendental integral function at

¹ cf. E. W. Hobson: The theory of functions of a real varirble (1907), p. 88.

most of the ρ^{th} order in z for any fixed value z' in $D(z')$: and so also for $\frac{\partial^N f(z, z')}{\partial z'^N}$.

As $a_i \neq a_j$ for $i \neq j$, we may assume that $|a_1| < |a_2| < |a_3| < \dots^1$. There are many transcendental integral functions of z , which have simple zero points at S and no others. Let $g(z)$ be one of the lowest order among them. Then the exponent of convergency of $|a_i|$, ($i=1, 2, 3, \dots$), is equal to the order of $g(z)$. As S is the set of the simple zero points of $g(z)$, $g'(a_i) \neq 0$, ($i=1, 2, 3, \dots$). Put

$$E_i(z, z') \equiv \left(\frac{1}{z-a_i} + \frac{1}{a_i} + \frac{z}{a_i^2} + \dots + \frac{z^{p_i-1}}{a_i^{p_i}} \right) \frac{P_i(z')}{g'(a_i)},$$

and if $a_1=0$, put

$$E_1(z, z') \equiv \frac{P_1(z')}{g'(o)} \cdot \frac{1}{z}.$$

If $\text{Max } |P_i(z')|$ in $D(z')$ be A_i , we have

$$\begin{aligned} |E_i(z, z')| &= \left| \left(\frac{1}{z-a_i} + \frac{1}{a_i} + \frac{z}{a_i^2} + \dots + \frac{z^{p_i-1}}{a_i^{p_i}} \right) \frac{P_i(z')}{g'(a_i)} \right| \\ &\leq \frac{1}{1 - \left| \frac{z}{a_i} \right|} \left| \frac{z^{p_i}}{a_i^{p_i+1}} \right| \cdot \left| \frac{A_i}{g'(a_i)} \right| \quad \text{for } |z| < |a_i|. \end{aligned}$$

Hence, we determine p_i so as to satisfy

$$\left| E_i(z, z') \right| \leq \varepsilon_i \quad \text{for } |z| \leq \frac{|a_i|}{2},$$

where $\varepsilon_i > 0$ and $\sum_{i=1}^{\infty} \varepsilon_i$ is convergent.

Then $\sum_{i=1}^{\infty} E_i(z, z')$ is absolutely and uniformly convergent in a field $E(z, z')$ (such that z' is any point in $D(z')$, while z is any finite point

¹ If the exponent of convergence of $|a_i|$, ($i=1, 2, 3, \dots$), be transfinite, we take a subset $S_1 = \{b_1, b_2, b_3, \dots\}$ of S , instead of S , for which $|b_i| < |b_j|$, ($i < j$), and the exponent of convergence of $|b_i|$, ($i=1, 2, 3, \dots$), is finite and is greater than ρ .

in the whole Gauss plane) except the vicinity of $z=a_i$, ($i=1, 2, 3, \dots$).

$$\text{Put } g(z) \sum_{i=1}^{\infty} E_i(z, z') \equiv F(z, z').$$

Then $F(z, z')$ is regular in $E(z, z')$ and takes the value $P_i(z')$ at $z=a_i$, ($i=1, 2, 3, \dots$). We now consider the function $f(z, z') - F(z, z')$. It is regular in $E(z, z')$ and vanishes at S . Hence we may put

$$f(z, z') - F(z, z') \equiv g(z) \cdot H(z, z')$$

where $H(z, z')$ is regular in $E(z, z')$, and is a transcendental integral function of z for any z' in $D(z')$.

$$\frac{\partial^N f(z, z')}{\partial z'^N} \equiv \frac{\partial^N F(z, z')}{\partial z'^N} + g(z) \cdot \frac{\partial^N H(z, z')}{\partial z'^N}.$$

$\frac{\partial^N f(z, z')}{\partial z'^N}$ is, as before, a transcendental integral function at most of the ρ^{th} order in z for any fixed value z' in $D(z')$. As $F(z, z')$ is a polynomial at most of the $(N-1)^{\text{th}}$ degree in z' , $\frac{\partial^N F(z, z')}{\partial z'^N} \equiv 0$. As $H(z, z')$ is regular in $E(z, z')$, and is a transcendental integral function of z for any z' in $D(z')$, $\frac{\partial^N H(z, z')}{\partial z'^N}$ must be one also. For if $\frac{\partial^N H(z, z')}{\partial z'^N} \equiv 0$, $\frac{\partial^N f(z, z')}{\partial z'^N}$ would be also identically zero, which is absurd. Thus from the identity

$$\frac{\partial^N f(z, z')}{\partial z'^N} \equiv g(z) \cdot \frac{\partial^N H(z, z')}{\partial z'^N},$$

we infer that the order of $g(z)$ can not exceed that of $\frac{\partial^N f(z, z')}{\partial z'^N}$, i. e., the exponent of convergency of $|a_i|$, ($i=1, 2, 3, \dots$), can not exceed ρ .
Q. E. D.

The theorem may be easily extended to the case where ρ is transfinite but less than $\omega^{\omega} = \mathcal{Q}$.

Studies on Camphor Oils II. On the Reaction of Japanese Acid Clay to Camphor White Oil.

By

Kashichi Ono.

(Received May 6, 1924)

At the time when this work was begun there could not be found any report of experiments made to investigate the action of Florida earth upon terpenes, except the work of Gurvitsch¹ who observed that pinene is thus transformed into camphene and polyterpenes.

Before the work was finished, however, valuable papers were published by C. S. Venable² and by K. Kobayashi³. Venable observed that by the action of Fuller's earth terpenes and unsaturated oxygenated terpenes are easily polymerized into dipinene and polyterpenes, while saturated oxygenated terpenes, cineol and p-cymene, do not undergo such polymerization, and that at a high temperature the dipinene so formed is slowly decomposed into low-boiling paraffin hydrocarbons and p-cymene. On distilling a mixture of turpentine oil or its main constituent, boiling at 158—161° with Japanese acid clay, K. Kobayashi and E. Yamaguchi obtained a distillate having a petroleum-like odour and a marked fluorescence similar to that of spindle oil prepared from natural petroleum. The distillate is considered by them to contain nearly equal quantities of aromatic and unsaturated hydrocarbons and

¹ Jour. Russ. Phys. Chem. Soc., **47**, 827 (1915); Jour. Chem. Soc., **108**, 933 (1915).

² Jour. Am. Chem. Soc., **45**, 728 (1923).

³ Kogyo-Kwagaku Zasshi (Jour. Chem. Soc. Tokyo) **26**, 463 (1923).

of naphtenic hydrocarbons.

1. ACTION OF JAPANESE ACID CLAY UPON CAMPHOR WHITE OIL.

The camphor white oil used in the experiment showed the following properties: Specific gravity 0.8756 (20°), refractive index 1.4698 (20°), optical rotation +20.45°, cineol content 22.30%, and from 200 c.c. of the oil it gave the following fractions when distilled:

Fraction	Distillate	d_{17}^*	$\alpha_D(1\text{dm.})$	n_D^{18}
	c.c.			
155—170°	10.7	0.8693	+ 10.50°	1.4708
170—175°	132.5	0.8752	+ 19.76°	1.4710
175—180°	38.2	0.8827	+ 26.56°	1.4705
above 180°	18.0	—	—	1.4931

General procedure:— Japanese acid clay previously dried at 110° was added to 400 grams of the camphor white oil in a copper flask (about 1 litre), and the mixture was gradually heated till the oil ceased to distill. The distillate was separated from water and dried with anhydrous sodium sulphate.

Japanese acid clay	Distillate	d	n_D
gr.	gr.		
100	353.2	0.8769(10°)	1.4926(12°)
200	345.3	0.8752(10°)	1.4930(11°)
300	341.0	0.8746(10°)	1.4928(11°)
*400	273.7	0.8436(10°)	1.4772(12°)
500	308.7	0.8678(20°)	1.4950(12°)
600	303.0	0.8569(10°)	1.4844(10°)

2. THE REACTION PRODUCTS.

400 grams of the camphor white oil were treated with 600 grams of Japanese acid clay in a manner analogous to that described

* This experiment was carried out in a glass flask.

in the general procedure. On repeating the same experiment 6 times 1820.3 grams of an oil (78.83% of the camphor oil) were obtained. This oil has a yellow colour with a strong greenish fluorescence and a petroleum-like odour. In order to remove terpenic and unsaturated hydrocarbons, the oil was first treated with a dilute permanganate solution and then with a concentrated sulphuric acid in cold, and finally was washed with caustic soda and water successively.

The refined oil thus obtained gave the following physical characters : d_{18}° 0.8531, $n_D^{18.5^{\circ}}$ 1.4751.

It was subjected to fractional distillation under atmospheric pressure and the following fractions were gathered..

Fraction	Distillate	Specific gravity	Refractive index
	gr.		(19°)
up to 100°	2.7	—	1.4015
100—110°	3.5	—	1.4084
110—120°	6.2	0.7363($\frac{20.5}{18}$)°	1.4138
120—130°	2.4	—	1.4200
130—140°	3.7	—	1.4222
140—150°	8.4	0.7649($\frac{20.5}{18}$)°	1.4287
150—160°	20.1	0.7823(")	1.4379
160—170°	63.7	0.8052(")	1.4503
170—180°	323.3	0.8243($\frac{20}{18}$)°	1.4616
180—190°	160.2	0.8327(")	1.4681
190—200°	61.3	0.8404($\frac{20.5}{18}$)°	1.4731
200—210°	6.9	0.8456(")	1.4759
210—220°	7.2	0.8502(")	1.4768
220—230°	10.6	0.8537(")	1.4771
230—240°	6.7	0.8598($\frac{18}{18}$)°	1.4790
240—250°	6.9	0.8678(")	1.4811
250—260°	7.3	0.8754(")	1.4849
260—270°	11.6	0.8854(")	1.4892
270—280°	9.8	0.8896(")	1.4920
280—290°	15.6	0.9028(")	1.4980
290—300°	34.0	0.9089(")	1.5026
300—310°	56.8	0.9184(")	1.5062
310—320°	57.0	0.9264(")	1.5096
320—330°	38.0	0.9285($\frac{20}{18}$)°	1.5116
330—340°	38.5	0.9286($\frac{19.5}{18}$)°	1.5150
340—350°	30.8	0.9341(")	1.5172
above 350°	6.8	—	—

1) Nitration of the fraction up to 100°

The fraction up to 100° was strongly nitrated, when a solid nitration product which was proved to be *m*-dinitrobenzene was produced.

2) Nitration of the fraction 110—120°

From the fraction boiling at 110—120° only a small quantity of a nitrated oil was produced, but it was not further examined. The main portion remaining inert was refined as before, and an oil having a refractive index of 1.4050 (23°), was obtained.

It was analyzed with the following results :

0.0938 gr. substance gave 0.2912 gr. CO₂ and 0.1145 gr. H₂O.

	Found	Cal.(C ₈ H ₁₆)
Carbon	84.68%	85.62%
Hydrogen	13.64	14.38

3) Nitration of the fraction 170—180°.

The well-cooled mixture of 320 grams of conc. nitric acid (sp. gr. 1.40) and 420 grams of conc. sulphuric acid (sp. gr. 1.48) was slowly added to 200 grams of the fraction 170—180° kept cooled in a flask. When the reaction was completed, the mixture was poured into a large amount of water, and the oily layers were separated.

The part unacted upon (about 95.2 grams) was washed with conc. sulphuric acid, fuming sulphuric acid, caustic soda, and water successively.

The colourless oil thus obtained gave the following physical properties: d_{4}^{20} 0.8050, n_D^{20} 1.4410.

In order to purify it, 60 c.c. of the oil were subjected to fractional distillation on metallic sodium 5 times under atmospheric pressure, and the following fractions were collected.

1st Distillation

Fraction	Distillate	d_{4}^{20}	n_D^{20}
	c.c.		
160—165°	5.9	0.7861	1.4366
165—171°	23.3	0.7969	1.4389
171—175°	16.2	0.8014	1.4400
175—180°	7.1	0.8038	1.4413
above 180°	5.7	—	1.4599

5th Distillation.

Fraction	Distil- late	$d(\frac{21}{4})^{\circ}$	$n_D^{21^{\circ}}$	Analysis					M. R*(Found)			Determination of Mol. wt.			
				Subst.	CO ₂	H ₂ O	C	H	C ₉ H ₁₈	C ₁₀ H ₂₀	C ₁₁ H ₂₂	Subst.	Benzene	Δt	Mol. wt.
155-160°	c.c. 1.9	—	1.4341	gr. —	—	gr. —	% —	% —	—	—	—	gr. —	gr. —	—	—
160-165°	6.0	0.7872	1.4360	0.1427	0.4487	0.1810	85.74	14.23	41.88	46.55	—	—	—	—	—
165-168°	5.0	0.7956	1.4384	0.0937	0.2921	0.1220	85.01	14.56	—	46.29	—	0.4075	26.5410	-0.556°	138.070
168-170°	18.6	0.8070	1.4396	0.0840 0.1537 0.1419	0.2620 0.4806 0.4434	0.1035 0.1954 0.1783	85.07 85.27 85.14	13.79 14.21 14.04	—	45.74	—	0.6307	26.4723	-0.861°	138.356
170-175°	9.6	0.8123	1.4408	0.2207	0.6892	0.3587	85.15	14.14	—	45.56	50.07	—	—	—	—
175-180°	2.1	—	1.4420	—	—	—	—	—	—	—	—	—	—	—	—
above 180°	0.4	—	1.4559	—	—	—	—	—	—	—	—	—	—	—	—

*Lorentz-Lorenz' formula

Molecular refraction, molecular weight and percentage of hydrogen and carbon calculated for $C_{10}H_{20}$ are as follows :

	$C_{10}H_{20}$
Carbon	85.62%
Hydrogen	14.38%
M. R (Eisenlohr's number)	46.18
Mol. wt.	140.160

As is seen in the foregoing table, the part unacted upon by a nitration mixture seems to consist of naphthenic hydrocarbons. The 3rd. and 4th. fractions may both be regarded as a hydrocarbon of the same composition but of different constitution, that is they would be isomers brought about by thermal action.

From the fact that all terpinic compounds existing in the camphor white oil have the hexamethylene ring, it seems very probable that the hydrocarbon $C_{10}H_{20}$ above obtained would be *p*-menthane. For confirmation, the substance was dehydrogenated by treating it with sulphur¹ and then oxidized with an alkaline potassium permanganate solution when *p*-oxyisopropyl benzoic acid, melting at 156°, was produced. It was analysed with the following results :

0.1374 gr. substance gave 0.3335 gr. CO_2 and 0.0765 gr. H_2O .

	Found	Cal. ($C_{10}H_{12}O_3$)
Carbon	66.19%	66.63%
Hydrogen	6.22	6.71

The nitrated oil (about 102.6 grams) after having been washed with water was distilled with superheated-steam, and the pale yellowish distillate was purified by fractional distillation. The fraction boiling at 114—117° under 5—6 m.m. (sp. gr. 1.0772 ($\frac{22}{4}$)^c refractive index 1.5367(22°) was collected as the main distillate.

0.1422 gr. substance gave 0.3533 gr. CO_2 and 0.0894 gr. H_2O .

	Found	Cal. ($C_{10}H_{13}NO_2$)
Carbon	67.75%	67.00%
Hydrogen :	7.03	7.31

¹ W. Friedmann : Jour Chem. Soc., 112, 13 (1917).

	Found	Cal. $C_{10}H_{13}NO_2\sqrt{3}$
Mol. refraction	51.785	51.039

Besides the above substance, a liquid of lighter specific gravity (about 10 grams) was also obtained as one of the distillation products, but its examination was omitted.

Oxidation of the nitro-compound.

Five grams of the nitro-compound and 100 c.c. of 25% caustic soda solution were mixed together, and to the mixture kept boiling on a sand bath a conc. potassium permanganate solution was added until the violet colour became permanent. Alcohol was then added to decolorize the violet colour, and precipitated manganese dioxide was filtered off. The alkaline filtrate thus obtained was acidified with hydrochloric acid, whereupon a small quantity of the white crystals of terephthalic acid separated out. It was filtered and the filtrate was extracted with ether. On distilling off the ether, the yellow oil which quickly crystallised into pale yellow crystals separated out. It was recrystallized with hot water when yellowish small needles, melting at 167—168° and having properties identical with o-nitro-p-oxyisopropyl benzoic acid were obtained.

It was analysed with the following results :

0.1648 gr. substance gave 0.3241 gr. CO_2 and 0.0745 gr. H_2O .

	Found	Cal. $(C_{10}H_{11}NO_5)$
Carbon	53.62%	53.31%
Hydrogen	5.05	4.92

Reduction of nitro-compound.

The nitro-compound was reduced with tin and hydrochloric acid, and the amino-compound obtained was analyzed as its hydrochloride.

0.2216 gr. the hydrochloride gave 0.1692 gr. AgCl

	Found	Cal. $(C_{10}H_{13}NH_2HCl)$
Chlorine	18.86%	19.10%

Its platinum double chloride was also analysed.

0.1238 gr. platinum double chloride gave 0.0343 gr. platinum.

	Found	Cal. $[(C_{10}H_{13}NH_2 \cdot HCl)_2PtCl_4]$
Platinum	27.70%	27.53%

From the above results it is beyond doubt that the nitro-compound is p-nitro-cymene ($C_6H_3CH_3 \cdot NO_2 \cdot C_3H_7$).

3. ACTION OF JAPANESE ACID CLAY UPON d-LIMONENE.

d-Limonene having the following physical properties was used in the experiment:

Boiling point $175-177^\circ$, specific gravity 0.8427(22°) refractive index 1.4736(22°), $n_D^{25} + 95.22$ (1 d.m.).

300 grams of d-limonene were treated with 400 grams of Japanese acid clay in a manner analogous to that described in the general procedure. The yield of the oil was about 222 grams.

200 c.c. of the oil thus formed (sp. gr. 0.8444(25°) and $n_D^{24} 1.4771$) were distilled under atmospheric pressure.

Fraction	Distillate	Refractive index
	c.c.	(23°)
up to 100°	0.7	1.4047
$100-110^\circ$	0.1	1.4116
$110-120^\circ$	0.4	1.4250
$120-130^\circ$	1.1	1.4275
$130-140^\circ$	1.3	1.4291
$140-150^\circ$	3.1	1.4357
$150-160^\circ$	10.0	1.4440
$160-170^\circ$	13.8	1.4529
$170-180^\circ$	67.2	1.4640
$180-190^\circ$	28.4	1.4700
$190-200^\circ$	9.7	1.4759
$200-210^\circ$	1.6	1.4786
$210-220^\circ$	1.9	1.4809
$220-230^\circ$	2.2	1.4851
$230-240^\circ$	3.5	1.4882
$240-250^\circ$	4.0	1.4971
$250-260^\circ$	2.5	1.5010
$260-270^\circ$	2.4	1.5087
$270-280^\circ$	2.7	1.5120
$280-290^\circ$	4.3	1.5169
$290-300^\circ$	6.2	1.5197
$300-310^\circ$	5.1	1.5212
$310-320^\circ$	7.6	1.5221
above 320°	19.4	—

Nitration of the fraction 170—180°.

70 grams of the fraction boiling at 170—180° (sp. gr. $0.8227(\frac{23}{4})^\circ$) were treated with a dilute permanganate solution and then nitrated.

The part remaining unacted upon (about 17 grams) gave the following physical properties:

Specific gravity $0.8036(\frac{23}{4})^\circ$, refractive index $1.4447(23^\circ)$.

13 c.c. of it were distilled on metallic sodium under atmospheric pressure.

1st Distillation

Fraction	Distillate	Refractive index
	c.c.	(24)°
160—165°	1.6	1.4359
165—170°	3.0	1.4372
170—175°	4.8	1.4394
175—180°	1.3	1.4410
above 180°	1.1	—

3rd Distillation

Fraction	Distillate	Refractive index
	c.c.	(24)°
159—160°	0.2	1.4344
160—165°	0.9	1.4359
165—168°	1.8	1.4376
168—170°	3.2	1.4380
170—175°	2.0	1.4400
175—180°	1.7	1.4414
above 180°	0.9	—

The fraction boiling at 168—170° [sp. gr. $0.7976(\frac{24}{4})^\circ$, mol. refraction* 46.13] was analyzed with the following results:

0.1319 gr. substance gave 0.4152 gr. CO₂ and 0.1638 gr. H₂O.

	Found	Cal. (C ₁₀ H ₂₀)
Carbon	85.84%	85.62%
Hydrogen	13.88	14.38

The nitrated compound [b. p. 113.5—117° (4—6 m.m.) and $d(\frac{23}{4})^\circ$ 1.0783] was oxidized with an alkaline potassium permanganate

* Mol. refraction calculated for C₁₀H₂₀ is 46.18.

solution and yellowish small needles, melting at 167—168° were obtained. It was identified to be o-nitro-p-oxyisopropyl benzoic acid.

4. ACTION OF JAPANESE ACID CLAY UPON CINEOL

300 grams of cineol (b. p. 175—177°, sp. gr. 0.9236(23°), n_D^{25} 1.4596, and $[\alpha]_D \pm 0$) were treated with 700 grams of Japanese acid clay, and about 206 grams of new oil were obtained.

200 grams of the oil (sp. gr. 0.8389(23°) and n_D^{25} 1.4735) were fractionated by distilling it under atmospheric pressure.

Fraction	Distillate	Refractive index
	gr.	(24)°
up to 100°	0.6	1.3937
100—110°	0.2	1.4112
110—120°	0.6	1.4188
120—130°	1.2	1.4271
130—140°	1.5	1.4345
140—150°	3.8	1.4411
150—160°	6.1	1.4492
160—170°	10.4	1.4551
170—180°	96.8	1.4641
180—190°	32.3	1.4716
190—200°	5.7	1.4770
200—210°	3.0	1.4800
210—220°	2.4	1.4821
220—230°	1.3	1.4850
230—240°	0.9	1.4890
240—250°	1.4	1.4931
250—260°	1.5	1.4990
260—270°	2.4	1.5038
270—280°	2.3	1.5082
280—290°	2.5	1.5115
290—300°	3.0	1.5159
300—310°	4.1	1.5179
310—320°	6.4	1.5183
above 320°	8.7	1.5272

70 grams of the fraction boiling at 170—180° (sp. gr. 0.8239 ($\frac{25}{4}$)°) were treated with a dilute permanganate solution and conc. sulphuric acid, and then nitrated. The nitro-compound (about 27 grams) having b. p. 115—119° (7—8 m.m.) and sp. gr. 1.0782($\frac{24}{4}$)°, was oxidized with an alkaline potassium permanganate solution and

o-nitro-p-oxyisopropyl benzoic acid, melting at 167—168° was obtained.

The part remaining unacted upon (about 11 grams) was also fractionated on distilling it on metallic sodium under atmospheric pressure.

1st Distillation

Fraction	Distillate	Refractive index
	c.c.	(25)°
160—165°	1.2	1.4360
165—170°	2.8	1.4374
170—175°	4.2	1.4391
175—180°	0.9	1.4401
above 180°	0.7	—

3rd Distillation

Fraction	Distillate	Refractive index
	c.c.	(25)°
up to 160°	0.1	1.4342
160—165°	0.3	1.4358
165—168°	1.2	1.4361
168—170°	2.7	1.4370
170—175°	1.6	1.4389
175—180°	1.3	1.4397
above 180°	0.8	—

The fraction boiling at 168—170° (sp. gr. 0.7942($\frac{15}{4}$)°, mol. refraction* 46.23) was analyzed with the following results:

0.1024 gr. substance gave 0.3197 gr. CO₂ and 0.1282 gr. H₂O.

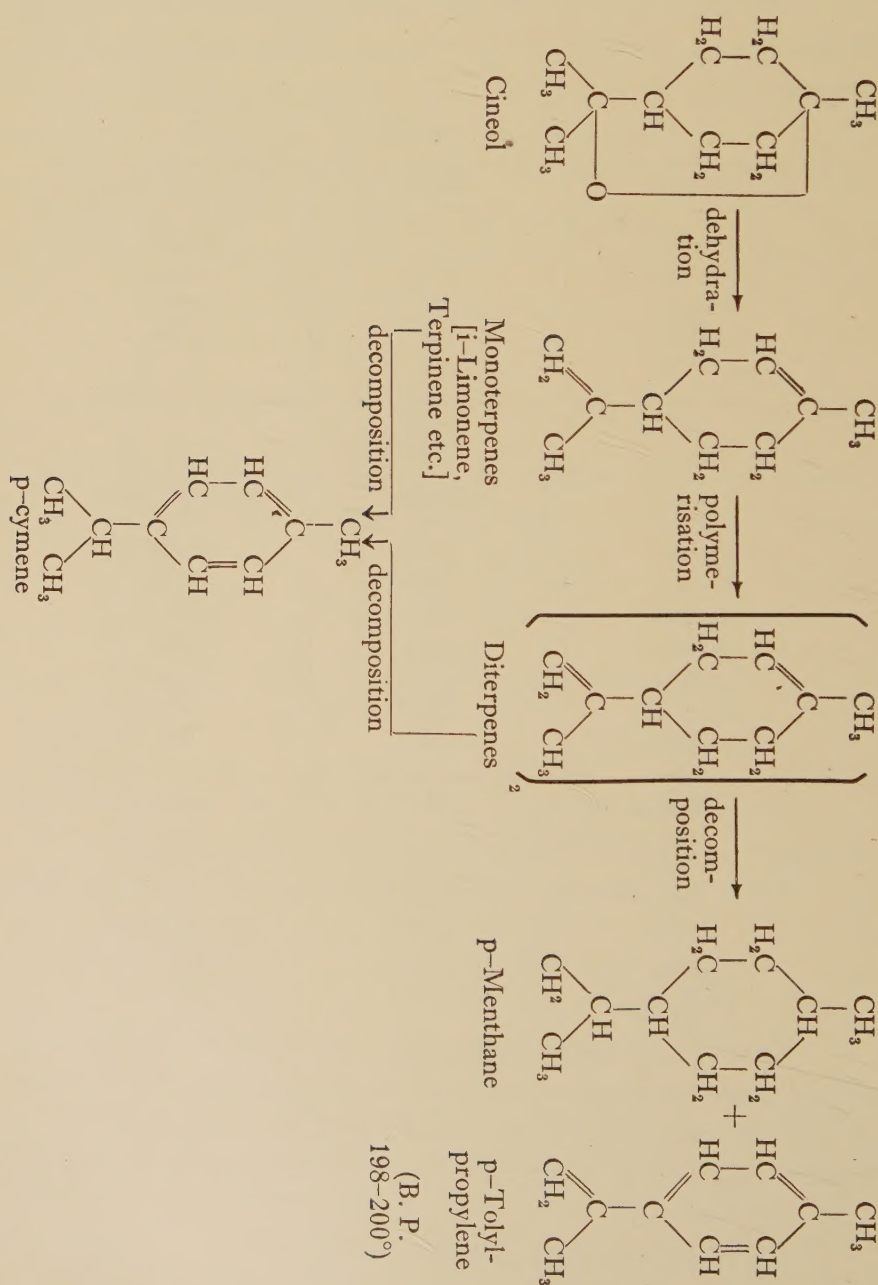
	Found	Cal. (C ₁₀ H ₂₀)
Carbon	85.13%	85.62%
Hydrogen	13.99	14.38

From these experimental results it appears evident that the main portion of cineol changes into p-cymene and p-menthane by the action of Japanese acid clay. These changes and also those observed by Venable¹ may all be looked upon as the results of dehydration and

* Mol. refraction calculated for C₁₀H₂₀ is 46.18.

¹ loc. cit.

polymerization which are in high temperatures followed by decomposition, as is represented in the following scheme :



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